

HOMOGENIZATION AND NON-HOMOGENIZATION OF CERTAIN NON-CONVEX HAMILTON-JACOBI EQUATIONS

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ABSTRACT. We continue the study of the homogenization of coercive non-convex Hamilton-Jacobi equations in random media identifying two general classes of Hamiltonians with very distinct behavior. For the first class there is no homogenization in a particular environment while for the second homogenization takes place in environments with finite range dependence. Motivated by the recent counter-example of Ziliotto [18], who constructed a coercive but non-convex Hamilton-Jacobi equation with stationary ergodic random potential field for which homogenization does not hold, we show that same happens for coercive Hamiltonians which have a strict saddle-point, a very local property. We also identify, based on the recent work of Armstrong and Cardaliaguet [1] on the homogenization of positively homogeneous random Hamiltonians in environments with finite range dependence, a new general class Hamiltonians, namely equations with uniformly strictly star-shaped sub-level sets, which homogenize.

1. INTRODUCTION

We continue the study of the homogenization of non-convex Hamilton-Jacobi equations in random media identifying two general classes of Hamiltonians with very distinct averaging behavior.

In particular we consider the asymptotic behavior, as $\varepsilon \rightarrow 0$, of the initial value problem

$$u_t^\varepsilon + H(Du^\varepsilon, \frac{x}{\varepsilon}) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad u(x, 0) = u_0(x), \quad (1.1)$$

with $H(\xi, x)$ a stationary ergodic random field in $x \in \mathbb{R}^d$ which is uniformly coercive in ξ but not convex.

A simple rescaling shows that the limiting behavior of the u^ε 's with initial datum $u(x, 0) = \xi \cdot x$ for some $\xi \in \mathbb{R}^d$ is the same as the long time average behavior of the solution u of the initial value problem

$$u_t + H(Du, x) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad u(x, 0) = \xi \cdot x. \quad (1.2)$$

Indeed the following is true for all $t > 0$:

$$\liminf_{\varepsilon \rightarrow 0} u^\varepsilon(0, t) = \liminf_{s \rightarrow \infty} \frac{u(0, s)}{s} \quad \text{and} \quad \limsup_{\varepsilon \rightarrow 0} u^\varepsilon(0, t) = \limsup_{s \rightarrow \infty} \frac{u(0, s)}{s}.$$

There is large body of work about the qualitative and quantitative homogenization of convex Hamilton-Jacobi equations in random media following respectively the papers of Souganidis [16], Rezakhanlou and Tarver [15], Lions and Souganidis [13, 14], Armstrong and Souganidis [4, 5], and Armstrong, Cardaliaguet and Souganidis [3].

In spite of this progress whether non-convex coercive Hamilton-Jacobi equations homogenize or not in random environments has been a long standing open question with some results available for Hamiltonians of very specific form; see Armstrong, Tran and Yu [6, 7] and Gao [10].

A significant step in the understanding of the problem is the recent work of Ziliotto [18], who constructed a non-convex Hamiltonian $h = h(\xi)$ and a stationary ergodic random field $V = V(x)$, such that (1.1) with Hamiltonian $H(\xi, x) = h(\xi) - V(x)$ does not homogenize for a particular linear initial datum.

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In the direction of homogenization the most recent progress is due to Armstrong and Cardaliaguet [1], who considered, among others, (1.2) in environments with finite range of dependence assumption in x and Hamiltonians $H(\xi, x)$ which are positively-homogeneous in the gradient variable ξ .

Our work builds upon these two contributions. In the non-homogenization direction we identify a general class of coercive nonlinearities h , namely ones with a strict saddle-point, which, when used in the separated Hamiltonian $H(\xi, x) = h(\xi) - V(x)$ with a potential V similar to the one in [18], give rise to non-homogenization. This result shows that non-homogenization may be a consequence of a local property of the nonlinearity. The outcome is also consistent with a conjecture by the second author that homogenization for non-convex Hamiltonians, in general, may fail, if the media obstructs the “characteristics” from reaching in time t all directions at distance $O(t)$.

In the direction of homogenization we are able to adapt the arguments of [1] to the class of Hamiltonians with star-shaped sub-level sets, which strictly includes Hamiltonians which are positively homogeneous in the gradient variable.

To state the result we first need make explicit the meaning of the strict-saddle point, which, of course, makes sense only for $d \geq 2$.

We say that a continuous function $h : \mathbb{R}^d \rightarrow \mathbb{R}$ has a strict saddle point at some $\xi_0 \in \mathbb{R}^d$, if there exist two non-trivial orthogonal subspaces P_1 and P_2 , which together span \mathbb{R}^d , and $\eta > 0$ so that for any unit vectors $e \in P_1$ and $f \in P_2$,

$$h(\xi_0 + \lambda e) < h(\xi_0) \text{ for } \lambda \in [-\eta, \eta] \setminus \{0\} \text{ and } h(\xi_0 + \lambda f) > h(\xi_0) \text{ for } \lambda \in [-\eta, \eta] \setminus \{0\}. \quad (1.3)$$

This is, for example, the case when h is C^2 at ξ_0 and has a strict saddle point in the usual calculus sense, that is

$$\begin{cases} Dh(\xi_0) = 0 \text{ and there exist } m \in \{1, \dots, d-1\} \text{ such that} \\ D^2h(\xi_0) \text{ has } m \text{ strictly positive eigenvalues and } d-m \text{ strictly negative eigenvalues.} \end{cases} \quad (1.4)$$

Of course (1.3) contains more general non-smooth examples, like the Hamiltonian of [18], and smooth examples like $h(\xi) = \xi_1^2 - \xi_2^4$ which has a strict saddle at the origin in \mathbb{R}^2 .

Our first result is:

Theorem 1.1. *Let $h : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous coercive Hamiltonian with a strict saddle point at some $\xi_0 \in \mathbb{R}^d$. There exist a stationary ergodic random field V such that the solution u^ε of (1.1) with Hamiltonian $H(\xi, x) = h(\xi) - V(x)$ and initial datum $u_0(x) = \xi_0 \cdot x$ does not homogenize in the sense that, for all $t > 0$,*

$$\liminf_{\varepsilon \rightarrow 0} u^\varepsilon(0, t) < \limsup_{\varepsilon \rightarrow 0} u^\varepsilon(0, t) \text{ almost surely.}$$

Actually we are able to relax the assumption that $Dh(\xi_0) = 0$ in (1.3), if we allow for space-time stationary ergodic Hamiltonians. The result is:

Corollary 1.2. *Let $h : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth, coercive with super-linear growth Hamiltonian and suppose that there is $\xi_0 \in \mathbb{R}^d$ such that, for some $1 \leq m \leq d-1$, $D^2h(\xi_0)$ has m strictly positive and $d-m$ strictly negative eigenvalues. There exists a space-time stationary ergodic random field $V = V(x, t)$ such that the solution u^ε to the initial value problem*

$$u_t^\varepsilon + h(Du^\varepsilon) - V\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) = 0 \text{ in } \mathbb{R}^d \times (0, \infty) \quad u(x, 0) = \xi_0 \cdot x$$

does not homogenize, in the sense that, for all $t > 0$,

$$\liminf_{\varepsilon \rightarrow 0} u^\varepsilon(0, t) < \limsup_{\varepsilon \rightarrow 0} u^\varepsilon(0, t) \text{ almost surely.}$$

Considering the homogenization result we show that the essential feature of the Hamiltonians used in [1] is not positive homogeneity in the gradient variable but rather star-shapedness of the sub-level sets.

A Hamiltonian H is said to have star-shaped sub-level sets (with respect to the origin) if

$$\text{for every } \mu \geq \inf_{\mathbb{R}^d} H(\cdot, x) \text{ and } x \in \mathbb{R}^d, \{ \xi : H(\xi, x) \leq \mu \} \text{ is strictly star-shaped with respect to } 0. \quad (1.5)$$

An easy generalization of our methods allows to consider Hamiltonians with each μ sub-level strictly star-shaped with respect to a different point ξ_μ (independent of x). Actually, since the method is quantitative in nature, it is necessary to quantify the strict star-shapedness of the sub-level sets; more precise details can be found in section 3.

The homogenization result we are obtaining here is an extension of [1] which studied positively homogeneous Hamiltonians and of [5] which studied quasi-convex Hamiltonians, that is H 's with convex sub-level sets.

A key step in establishing homogenization is to understand the limiting behavior of the approximate correctors $v^\delta(\cdot; \xi)$ for $\xi \in \mathbb{R}^d$, that is the solutions of the problem,

$$\delta v^\delta + H(\xi + Dv^\delta, x) = 0 \quad \text{in } \mathbb{R}^d.$$

In view of the assumed coercivity, homogenization follows (see, for example, [5, 13, 14]) if it is shown that

$$-\delta v^\delta(0, \xi) \rightarrow \overline{H}(\xi) \quad \text{almost surely for every } \xi \in \mathbb{R}^d. \quad (1.6)$$

We state our homogenization result as a rate of convergence for (1.6); the statement here is rather vague and is weaker than the result we actually prove since we are avoiding stating the technical assumptions in the introduction. The full result can be found in Proposition 3.13.

Theorem 1.3. *If H is a finite range of dependence stationary random field of Hamiltonians satisfying standard coercivity and continuity assumptions and (a quantitative version of) (1.5), then, there exists a modulus ρ and constants $C = C(H(\xi)) > 0$ and $c = c(H(\xi)) > 0$ such that, for every $\delta, \lambda > 0$,*

$$\mathbb{P}(|\delta v^\delta(0, \xi) + \overline{H}(\xi)| > \lambda \delta \rho(\delta |\log \delta|)) \leq C \exp(-c\lambda^{4 \wedge d}).$$

The approach in this paper, as in [1, 3, 5], is not to study the approximate corrector problem directly but instead to focus on the so called metric problems associated with the Hamiltonian.

In particular, given a plane with unit direction e , we study the asymptotic properties of the non-negative solution m_μ of the metric problem to the plane

$$H(Dm_\mu, x) = \mu \quad \text{in } \{x \cdot e > 0\} \quad \text{with } m_\mu = 0 \quad \text{on } \mathcal{H}_e := \{y \in \mathbb{R}^d : y \cdot e = 0\}. \quad (1.7)$$

In the context of level set motions this would perhaps be better called the arrival time problem since, in that case, the solution $m_\mu(x)$ corresponds to the time that a front, which started from the plane \mathcal{H}_e , reaches the point x . Then the level sets of $\{y \in \mathbb{R}^d : m_\mu(y) = t\}$ correspond to the locations of the propagating front at times $t > 0$. At first glance this interpretation seems to be limited to positively 1-homogeneous level set motion type Hamiltonians. However, we show here that metric problems for Hamiltonians with star-shaped sub-levels can be transformed to metric problems for level-set motion type equations.

Although we are doubtful that the star-shaped assumption is necessary for homogenization, we explain next why it is (in a sense) necessary for a method like that of [1], which is based on the solutions of the planar metric problem.

The star-shapedness assumption guarantees uniqueness for metric problems like (1.7). As a matter of fact, for spatially homogeneous Hamiltonians, it is indeed the sharp assumption for uniqueness. With x dependence, it is not clear what is the correct assumption for uniqueness of the planar metric problems.

In fact uniqueness for the associated metric problems is not necessary to obtain a homogenized Hamiltonian. For example, [6] deals with this non-uniqueness of the metric problem solutions for Hamiltonians of the form

$$H(\xi, x) = \Psi(h(\xi)) - V(x),$$

with $h \geq 0$ convex and coercive and $\Psi : [0, \infty) \rightarrow \mathbb{R}$ smooth such that $\lim_{s \rightarrow \infty} \Psi(s) = \infty$. Speaking very informally in [6] the metric problem $\Psi(h(Du)) - V(x) = \mu$ is replaced by $h(Du) = \Psi^{-1}(\mu + V(x))$, where Ψ^{-1} is multi-valued. It turns out, however, that in each continuous branch of Ψ^{-1} , the metric problems for h have unique solutions.

Conceivably there might exist some more general method of parametrizing the non-unique solutions of the planar metric problem as solutions of sub-problems with a more amenable Hamiltonian, however this seems quite difficult to realize.

We remark that, in view of our conclusions, the homogenization results in [6] are rather unstable. For example, it is easy to see that arbitrary small perturbations of $H(p, x) = (|p|^2 - 1)^2 - V(x)$, a typical problem studied in [6], have strict-saddle points and, hence, homogenization fails.

Organization of the paper. The paper is divided into two parts dealing separately with the non-homogenization and homogenization results. Section 2, which is about non-homogenization, is subdivided itself into several subsections. First we give the set up and precise assumptions on the Hamiltonian and the random field (subsection 2.1), then in subsection 2.2 we construct barriers which imply non-homogenization, subsections 2.3 and 2.4 are about the construction of the random field and its properties. Subsection 2.5 is about a mixing property satisfied by the field, and, finally, subsection 2.6 is about the space-time result. Section 3 is dedicated to the homogenization result and is also divided in several subsections. In subsection 3.1 we give a general outline of the proof of quantitative homogenization developed in [1, 3]. In subsection 3.2 we give the specific assumptions on the Hamiltonian, explain more rigorously the role of star-shapedness, and discuss possible generalizations. In subsection 3.3 we explain the reduction to the 1-positively homogeneous metric problem, and in subsection 3.4 we introduce the planar problem. In subsections 3.5-3.6 we go through the quantitative homogenization proof in more detail and, in particular, the fluctuations estimate and the convergence of the expectations (bias estimate), giving detailed proofs in the places where the star-shapedness condition comes in. The effective equation is introduced in subsection 3.7 where we also describe some of its properties, and, finally, in subsection 3.8 we touch upon the relationship with the approximate problem

Notation. We work in \mathbb{R}^d with the Euclidian metric d , and S^{d-1} and $B_r(x)$ are respectively the unit sphere and the open ball in \mathbb{R}^d centered at x with radius r and $B_r := B_r(0)$. For $x \in \mathbb{R}^d$, $|x|_\infty := \max_{i \in \{1, \dots, m\}} |x_i|$. Given $a, b \in \mathbb{R}$, $a \vee b := \max(a, b)$ and $a \wedge b := \min(a, b)$. We write A^C for the complement of the set A . For $A, B \subset \mathbb{R}^d$ compact, we denote by $d(A, B)$ is their Hausdorff distance. If \mathcal{A} is a collection of subsets of a certain set Ω , $\sigma(\mathcal{A})$ is the smallest σ -algebra on Ω containing \mathcal{A} . Given two quantities A and B , $A \lesssim B$ means that there exists some $C > 0$ depending only in the data such $A \leq CB$. If there $C_1 B \leq A \leq C_2 B$ we write $A \approx B$. Finally, $[x]$ denotes the integer part of x , and, for $p \in \mathbb{R}^d \setminus \{0\}$, $\hat{p} : p/|p|$.

Terminology. A constant C is said to depend on the data and we write $C(\text{data})$, if it depends only the several constants in the assumptions. Throughout the paper, unless otherwise said, solutions should be interpreted as the Crandall-Lions viscosity solutions.

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2. NON-HOMOGENIZATION FOR HAMILTONIANS WITH A STRICT SADDLE POINT

2.1. Assumptions and results. We consider Hamiltonians of the form

$$H(\xi, x) = h(\xi) - V(x). \quad (2.1)$$

In addition to (1.3) we assume that

$$h \text{ is coercive,} \quad (2.2)$$

and

$$V : \mathbb{R}^d \rightarrow [-1, 1] \text{ is 2-Lipschitz stationary ergodic random field on } \mathbb{R}^d. \quad (2.3)$$

Although we are yet not introducing here the probability space, we will freely write $\omega \in \Omega$.

We show that the potential V constructed in [18] can be used to prove non-homogenization for this more general class of Hamiltonians; recall that [18] works with a very specific h .

We point out that, if (1.3) holds for some η , then it holds for all smaller η . Thus the non-homogenization behavior is due to the random field and the local behavior near a generic saddle point of the Hamiltonian and not the global behavior of the specific h used in [18].

Since the arguments for $d \geq 3$ are similar to the ones when $d = 2$, here, to simplify the presentation, we assume that $d = 2$.

Then (1.3) has the following simpler form: There exist two orthogonal unit directions e_1 and e_2 and $\eta > 0$ such that, for $s \in [-\eta, \eta] \setminus \{0\}$,

$$h(\xi_0 + se_1) < h(\xi_0) < h(\xi_0 + se_2). \quad (2.4)$$

To further simplify the presentation and the arguments, below we make some more deductions from assumption (2.4). After a rotation, we can assume that e_1 and e_2 are the axis directions. Adding a constant and using $\lambda^{-1}h(\xi_0 + \eta^{-1}\xi)$ in place of h , we can assume that $h(\xi_0) = 0$, $\xi_0 = 0$ and $\eta = 1$. Finally, by selecting λ appropriately, we have

$$0 < 3 \leq \min\{-h(\pm e_1), h(\pm e_2)\}.$$

The continuity of h yields that, for some small $\delta > 0$ and $c_\delta > 0$,

$$\begin{cases} h(\xi) \leq -2 & \text{in } B_\delta(\pm e_1), \quad h(\xi) \geq 2 & \text{in } B_\delta(\pm e_2), \text{ and} \\ h(\xi) \leq c_\delta & \text{in } |\xi_2| \leq \delta, |\xi_1| \leq 1 \text{ and } h(\xi) \geq -c_\delta & \text{in } |\xi_1| \leq \delta, |\xi_2| \leq 1. \end{cases} \quad (2.5)$$

We will assume that we have a random field V which looks – vis-à-vis the Hamiltonian h – qualitatively different at infinitely many different dyadic time/length scales $T_n = 2^n$.

More precisely, we assume that V has the following two properties:

Property 1. For every $\theta > 0$ and almost surely, there exists a sequence $n_k(\omega) \rightarrow \infty$ such that, for all $k \geq 1$, $V(x) = -1$ on a horizontal line segment of length $\frac{4}{\delta}T_{n_k}$ centered at distance from the origin smaller than $[\theta T_{n_k}]$.

Property 2. For every $\theta > 0$ and almost surely, there exists a sequence $n'_k(\omega) \rightarrow \infty$ such that, for all $k \geq 1$, $V(x) = 1$ on a vertical line segment of length $\frac{4}{\delta}T_{n'_k}$ centered at distance smaller than $[\theta T_{n'_k}]$.

In subsection 2.3 and 2.4 we construct a random field satisfying (2.3) and Properties 1 and 2. For the motivation, coming from the differential game interpretation of the Hamilton-Jacobi equation, see [18].

In view of the remark at the beginning of the introduction and the simplifications above we study the long time average of the solutions to the initial value problem

$$u_t + h(Du) - V(x) = 0 \text{ in } \mathbb{R}^d \times (0, \infty) \text{ with } u(\cdot, 0) \equiv 0. \quad (2.6)$$

We remind that the choice of initial data is due to our normalization $\xi_0 = 0$.

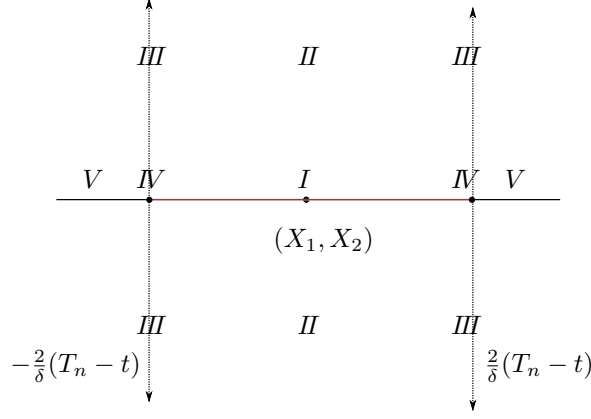


FIGURE 1. The domain of the barrier function v_+ is split up into regions.

Proposition 2.1. *Assume (1.3) and (2.2) and let u be the solution of (2.6) with random potential field V satisfying (2.3) and Properties 1 and 2. Then, almost surely,*

$$\liminf_{t \rightarrow \infty} \frac{u(0, t)}{t} \leq -1 + c_\delta \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{u(0, t)}{t} \geq 1 - c_\delta. \quad (2.7)$$

2.2. The proof of Proposition 2.1 and the construction of the barriers. Since the arguments are symmetric, here we only prove the first part of (2.7). The idea of the proof is that the potential V creates “obstacles” for the propagation along vertical and horizontal directions, thus “obstructing” the averaging. The obstacles are quantified by special super- and sub-solutions, that is barriers, to (2.6). This comes to play by the existence of special sub- and super-solutions that force the claimed inequality between the largest and smallest possible long time averaged limits.

The proof of Proposition 2.7: Fix $\theta > 0$, $\omega \in \Omega$ and $n \geq 1$ so that Property 1 holds. Then $V = -1$ on a horizontal segment of length $\frac{4}{\delta}T_n$ centered at $X = (X_1, X_2)$ with $|X| \leq \theta T_n$.

It turns out, as it is shown below, that

$$v_+(x, t) := -(1 - c_\delta)t + |x_2 - X_2| + (\delta|x_1 - X_1| + 2(t - T_n))_+$$

is a super-solution to

$$v_t + H(Dv, x) \geq 0 \quad \text{in } \mathbb{R}^2 \times (0, T_n) \quad \text{and} \quad v(x, 0) \geq 0. \quad (2.8)$$

It follows that

$$\frac{u(0, T_n)}{T_n} \leq \frac{v_+(0, T_n)}{T_n} \leq \frac{-(1 - c_\delta)T_n + (1 + \delta)\theta T_n}{T_n} \leq -1 + c_\delta + (1 + \delta)\varepsilon. \quad (2.9)$$

Since there are almost surely infinitely many T_n as above

$$\liminf_{t \rightarrow \infty} \frac{u(0, t)}{t} \leq -1 + c_\delta + (1 + \delta)\theta \quad \text{almost surely.}$$

Letting $\theta \rightarrow 0$ yields

$$\liminf_{t \rightarrow \infty} \frac{u(0, t)}{t} \leq -1 + c_\delta \quad \text{almost surely.}$$

Next we check that v_+ satisfies (2.8). That $v(x, 0) \geq 0$ for all $x \in \mathbb{R}^2$ is immediate.

For the super-solution property in $\mathbb{R}^2 \times (0, T_n)$, we need to identify the sub-differential of v_+ ,

$$D^-v_+(x, t) = D_x^-v_+(x, t) \times \partial_t^-v_+(x, t).$$

Recall that, for a continuous $f : \mathbb{R}^m \rightarrow \mathbb{R}$, the sub-differential $D^-f(z)$ is

$$D^-f(z) := \{p \in \mathbb{R}^m : f(z') \geq f(z) + p \cdot (z' - z) + o(|z' - z|)\}.$$

We split $\mathbb{R}^2 \times (0, T_n)$ into five subregions depending on the location of $(x, t) \in \mathbb{R}^2 \times (0, T_n)$; see Figure 2.2 for reference.

Region I: The line segment $x_2 = X_2$, $|x_1 - X_1| < \frac{2}{\delta}(T_n - t)$ and $t \in (0, T_n)$. Here $V(x) = -1$ and

$$D_x^-v_+(x, t) = \{0\} \times [-1, 1] \quad \text{and} \quad \partial_t v_+(x, t) = -1 + c_\delta.$$

Then (2.5) yields the super-solution condition,

$$-1 + c_\delta + h(\xi) - V(x) = h(\xi) + c_\delta \geq 0 \quad \text{for } \xi \in \{0\} \times [-1, 1].$$

Region II: Off the line segment, $x_2 \neq X_2$ and $|x_1 - X_1| < \frac{2}{\delta}(T_n - t)$ and $t \in (0, T_n)$. Here

$$D_x v_+(x, t) = \text{sgn}(x_2 - X_2)e_2 \quad \text{and} \quad \partial_t v_+ = -1 + c_\delta.$$

Since $V \leq 1$ on \mathbb{R}^d , it follows that,

$$\partial_t v_+ + H(D_x v_+, x) = -1 + c_\delta + h(\pm e_2) - V(x) \geq -1 + 3 - V(x) \geq 3 - 2 = 1 > 0.$$

Region III: The vertical lines $|x_1 - X_1| = \frac{2}{\delta}(T_n - t)$ except for $x_2 = X_2$ and $t \in (0, T_n)$. We have

$$D^-v_+(x, t) \subseteq [-\delta, \delta] \times \{1, -1\} \quad \text{and} \quad \partial_t^-v_+(x, t) \subseteq [-1, 1 + c_\delta].$$

The choice of δ yields that $h(\xi) \geq 2$ for $\xi \in [-\delta, \delta] \times \{1, -1\}$. It follows that, for every $(\xi, s) \in D^-v_+(x, t)$,

$$s + H(\xi, x) \geq -1 + h(\xi) - V(x) \geq h(\xi) - 2 \geq 2 - 2 = 0.$$

Region IV: The intersection of the horizontal segment $x_2 = X_2$ and the vertical lines $|x_1 - X_1| = \frac{2}{\delta}(T_n - t)$ for all $t \in (0, T_n)$. We have $V(x) = -1$ and

$$D_x^-v_+(x, t) = [-\delta, \delta] \times [-1, 1] \quad \text{and} \quad \partial_t^-v_+(x, t) = [-1 + c_\delta, 1 + c_\delta].$$

Now (2.5) gives $h(\xi) \geq -c_\delta$ for all $\xi \in D_x^-v_+(x, t)$, and, for every $(\xi, s) \in D^-v_+(x, t)$, we get the supersolution condition

$$s + H(\xi, x) \geq -1 + c_\delta - c_\delta - V(x) = 0,$$

and, finally,

Region V: The region $x \in \{(y_1, y_2) \in \mathbb{R}^2 : |y_1 - X_1| > \frac{2}{\delta}(T_n - t)\}$ for all $t \in (0, T_n)$. We know that $V(x) \leq 1$ and,

$$D_x^-v_+(x, t) \subseteq [-\delta, \delta] \times [-1, 1] \quad \text{and} \quad \partial_t^-v_+(x, t) = \{1 + c_\delta\}.$$

Since (2.5) gives $h(\xi) \geq -c_\delta$ for all $\xi \in D_x^-v_+(x, t)$, for every $(\xi, s) \in D^-v_+(x, t)$,

$$s + H(\xi, x) \geq 1 + c_\delta - c_\delta - V(x) \geq 1 - V(x) \geq 0.$$

2.3. The construction of the random field. We present here the construction of the random field V in \mathbb{R}^d following the ideas in [18].

We fix an intermediate dimension $m \in \{1, \dots, d-1\}$ and differentiate between the m -dimensional “horizontal” subspace spanned by e_1, \dots, e_m and the $d-m$ -dimensional “vertical” subspace spanned by e_{m+1}, \dots, e_d , and, for $x \in \mathbb{R}^d$, we set $x^1 := x_1 e_1 + \dots + x_m e_m$ and $x^2 := x_{m+1} e_{m+1} + \dots + x_d e_d$.

For each $k \in \mathbb{Z}^d$, let X_k, Y_k be i.i.d. dyadic, that is $\mathbb{D} := \{2^n : n \in \mathbb{N}\}$ valued, random variables with distribution, for each n , $\mathbb{P}(X_k = 2^n) = \alpha_d 2^{-dn}$ with $\alpha_d = (2^d - 1)$.

For concreteness we take the probability space to be $\Omega := (\mathbb{D} \times \mathbb{D})^{\mathbb{Z}^d}$ and denote by \mathcal{F} the σ -algebra generated by cylinder sets.

We give an informal description of the construction, which will then be made precise. Consider placing at each $k \in \mathbb{Z}^d$ a “horizontal” cube (parallel to $\mathbb{R}^m \times \{0\}^{d-m}$) centered at k of side length X_k and a “vertical” cube (parallel to $\{0\}^m \times \mathbb{R}^{d-m}$) centered at k of side length Y_k . Now we remove some of these cubes so that there are no intersections between horizontal and vertical ones. We go through each of the

X_k and Y_k (the order is irrelevant), and, if the horizontal (resp. vertical) cube centered at k intersects any vertical (resp. horizontal) cube with larger (or equal) side length, we “mark it” for deletion. After going through all the horizontal and vertical cubes and making the required markings, we then remove all the cubes which have been marked for deletion. The remaining configuration of cubes will have no intersections between horizontal and vertical cubes.

Now we make this construction precise. We assign to each $k \in \mathbb{Z}^d$ the mark $M_k \in \{-1, 0, 1\}$ according to the following rules:

$$M_k = -1 \quad \text{if } Y_\ell < \max\{X_k, 2|(\ell - k)^2|_\infty\}, \text{ for all } \ell \in \mathbb{Z}^d \text{ with } |(\ell - k)^1|_\infty \leq X_k/2, \quad (2.10)$$

$$M_k = 1 \quad \text{if } X_\ell < \max\{Y_k, 2|(\ell - k)^1|_\infty\}, \text{ for all } \ell \in \mathbb{Z}^d \text{ with } |(\ell - k)^2|_\infty \leq Y_k/2. \quad (2.11)$$

$$M_k = 0 \quad \text{if neither of the two alternatives in (2.10) and (2.11) hold.} \quad (2.12)$$

In terms of the informal description, above $M_k = -1$ (resp. $M_k = 1$) signifies that the horizontal (resp. vertical) cube centered at k is not deleted, while $M_k = 0$ signifies that both the horizontal and vertical cubes centered at k are deleted. Note that (2.10) and (2.11) are disjoint events, since they are subsets, respectively, of the disjoint events $\{X_k > Y_k\}$ and $\{Y_k > X_k\}$. Moreover, the computations of the following section will verify that $\mathbb{P}(M_k = \pm 1) > 0$, that is the construction results in something non-trivial.

If $M_k = -1$ (resp. $M_k = 1$), we assign to $k \in \mathbb{Z}^d$ a horizontal (resp. vertical) cube H_k (resp. R_k) of side length X_k (resp. Y_k). We denote by H (resp. R) the union of all such H_k (resp. R_k). We set

$$V_-(x) := \min\{-1 + 2d(x, H), 0\} \quad \text{and} \quad V_+(x) := \max\{1 - 2d(x, R), 0\}.$$

It follows from the definition of M_k that $d(R, H) \geq 1$, and, hence, the sets $\{V_+ > 0\}$ and $\{V_- < 0\}$ are disjoint.

The potential V is then defined

$$V(x) := V_+(x) + V_-(x).$$

The random field V constructed above is stationary with respect to \mathbb{Z}^d -translations. Moreover, since \mathbb{P} is a product measure, the \mathbb{Z}^d translation action on Ω is ergodic. Later (subsection 2.6) we will need to construct a random field which is stationary with respect to \mathbb{R}^d translations and ergodic. This is a standard construction; see, for example, the book of Jikov, Kozlov and Oleinik [11]. The translation τ is taken to be a uniform random point of $[0, 1)^d$, which is independent of X_k and Y_k , and $\bar{V}(x) = V(x - \tau)$. This new random field is \mathbb{R}^d -stationary and remains ergodic. We note that this construction does not preserve strong mixing, a limitation which is relevant later in the paper.

2.4. The properties of the random field. We show that the random field constructed in the previous subsection satisfies Property 1 and Property 2. Since they are symmetric, we will only check Property 1. Although we follow some of the arguments in [18], it is, however, necessary to use extra care when applying to converse of the Borel-Cantelli Lemma since the events involved are not independent.

We take $T_n := 2^n$ and let $0 < \theta \leq \frac{1}{2}$ be a dyadic rational. The events B_n related to Property 1 are

$$B_n := \{\exists k \in \mathbb{Z}^d \text{ with } |k| \leq \theta T_n \text{ and } V = -1 \text{ on an } m\text{-cube centered at } k \text{ of side length } \geq T_n\}.$$

The goal is to show that B_n occurs infinitely often. Since they are not independent, it is not enough to show that $\liminf \mathbb{P}(B_n) > 0$. Instead we need to use a converse version of the Borel-Cantelli Lemma for non-independent sets; see, for example, Bruss [9] and Tasche [17]. This requires to define the notion of α -mixing.

Definition 2.1. Let $(Z_n)_{n \in \mathbb{N}}$ be a sequence of random variables on a common probability space. The α -mixing coefficients are defined by

$$\alpha(n) := \sup\{|\mathbb{P}(E \cap F) - \mathbb{P}(E)\mathbb{P}(F)| : k \in \mathbb{N}, E \in \sigma(Z_1, \dots, Z_k) \text{ and } F \in \sigma(Z_{k+n})\}.$$

If $\alpha(n) \rightarrow 0$ as $n \rightarrow \infty$, the sequence Z_n is called α -mixing.

The result we use is the following.

Theorem 2.2 ([17]). *Let $(A_k)_{k \in \mathbb{N}}$ be a sequence of events with non-increasing probabilities $\mathbb{P}(A_k)$. Then $\mathbb{P}(A_k \text{ infinitely often}) = 1$ is implied by either of the following two conditions:*

(i) *There exists $r \in [-1, \infty)$ such that*

$$\sum_{n=1}^{\infty} n^{-\frac{1}{r+2}} \mathbb{P}(A_n) = +\infty \quad \text{and} \quad \sum_{n=1}^{\infty} n^r \alpha(n) < \infty.$$

(ii) *There exists $b \in (0, 1)$ such that,*

$$\sum_{n=1}^{\infty} \frac{\mathbb{P}(A_n)}{1 + \log n} = +\infty \quad \text{and} \quad \alpha(n) = O(b^n).$$

Next we formulate, in terms of the random variables X_k^j , a sufficient condition for the event B_n to occur. For this, it is convenient to assume that $n \geq n_\theta$ sufficiently large so that $\theta T_{n-1} \geq 1$. The claim is that, in order for B_n to occur, it suffices that

$$\begin{cases} (1) \text{ one of the } X_k \text{'s with } |k| \leq \theta T_n \text{ has value } T_n \\ \text{and} \\ (2) \text{ for every } |k^1|_\infty \leq T_n \quad Y_k < \max\{T_n, |k^2|_\infty\}. \end{cases}$$

Note that if (1) holds, there exists $|k_0| \leq \theta T_n$ with $X_{k_0} = T_n$. For B_n to occur, we only need to check that $M_{k_0} = -1$. Indeed, since $\theta \leq 1/2$, and, hence, $\theta/(1-\theta) \leq 1$, the fact that (2) holds for every $|k^1|_\infty \leq T_n$, and, thus, for every $|(k - k_0)^1|_\infty \leq \frac{1}{2}T_n$, yields

$$\begin{aligned} Y_k &< \max\{T_n, |k^2|_\infty\} \leq \max\{T_n, |(k - k_0)^2|_\infty + |k_0^2|_\infty\} \leq \max\{T_n, |(k - k_0)^2|_\infty + \theta T_n\} \\ &\leq \max\{T_n, (1 + \frac{\epsilon}{1-\epsilon})|(k - k_0)^2|_\infty\} \leq \max\{T_n, 2|(k - k_0)^2|_\infty\}. \end{aligned}$$

Dealing with condition (2) is more delicate and we discuss this later.

Next we consider the events C_n and D_n which formalize conditions (1) and (2) respectively and together guarantee the occurrence of B_n . For this, it is convenient to incorporate some independence in the C_n .

We have:

$$C_n := \{\exists k \in \mathbb{Z}^d \text{ with } \theta T_{n-1} < |k|_\infty \leq \theta T_n \text{ and } X_k = T_n\}, \quad (2.13)$$

and

$$D_n := \{\forall k \in \mathbb{Z}^d \text{ with } |k^1|_\infty \leq T_n \text{ it holds } Y_k < \max\{T_n, |k^2|_\infty\}\}. \quad (2.14)$$

Note that the restriction $\theta T_{n-1} < |k|_\infty \leq \theta T_n$ in C_n makes the C_n independent in n .

The following lower bounds on the probability of C_n and D_n are the same as in [18]. The proof is presented later in the subsection.

Lemma 2.3. *Let C_n and D_n be defined as above. Then*

$$\inf_{n \geq n_\theta} \mathbb{P}(C_n), \inf_{n \geq n_\theta} \mathbb{P}(D_n) > 0 \quad \text{and} \quad \inf_{n \geq n_\theta} \mathbb{P}(C_n \cap D_n) > 0.$$

Just the lower bounds are not sufficient to apply the converse Borel-Cantelli and conclude that, almost surely, $C_n \cap D_n$ occurs infinitely often. The events C_n are mutually independent, since they are defined in terms of disjoint collections of the X_k . The same is true for $\{C_n\}_{n=1}^\infty, D_j\}$, since the Y_k and X_k are independent of each other. This is not, however, the case for the D_n 's and to conclude we need to establish the α -mixing property discussed earlier in the subsection.

Lemma 2.4. *Let $n, \ell \in \mathbb{N}$. For any $E \in \sigma(D_1, \dots, D_\ell)$ and $F \in \sigma(D_{\ell+n})$,*

$$|\mathbb{P}(E \cap F) - \mathbb{P}(E)\mathbb{P}(F)| \leq C_{d,m} 2^{-nm}.$$

The mutual independence of the $(C_n)_{n=1}^\infty$ and their independence from the D_n yield the same α -mixing property for the sequence $C_n \cap D_n$.

It is immediate from Lemma 2.3 that

$$\sum_{n=1}^{\infty} \frac{\mathbb{P}(C_n \cap D_n)}{1 + \log n} = +\infty.$$

We remark that there is some leeway here to allow for some decay of $\mathbb{P}(C_n \cap D_n)$ while maintaining that the sum is infinite. Since, in view of Lemma 2.4, the events $C_n \cap D_n$ are α -mixing with exponential tails, we can apply Theorem 2.2 (ii) to conclude that

$$\mathbb{P}(B_n \text{ infinitely often}) \geq \mathbb{P}(C_n \cap D_n \text{ infinitely often}) = 1. \quad (2.15)$$

Proof of Lemma 2.4. We claim that $E \in \sigma(D_{\ell+n})$ and $F \in \sigma(D_1, \dots, D_\ell)$ are conditionally independent on $\sigma(I_{\ell,n})$, where $I_{\ell,n}$ is the event

$$I_{\ell,n} := \{\forall k \in \mathbb{Z}^d \text{ with } |k^1|_\infty \leq T_\ell \text{ and } Y_k < \max\{T_{\ell+n}, |k^2|_\infty\}\}$$

which has probability close to 1.

First we note that the facts that $D_j \in \sigma((Y_k)_{|k^1|_\infty \leq T_j})$ and the Y_k 's are i.i.d. imply that D_1, \dots, D_ℓ are independent of $(Y_k)_{|k^1|_\infty > T_\ell}$.

To check the conditional independence, we first compute $\mathbb{P}(E, F | I_{\ell,n})$ with $E = D_n$ and for this we define

$$D_{\ell,n} := \{Y_k < \max\{T_{\ell+n}, |k^2|_\infty\} \text{ holds for all } k \in \mathbb{Z}^d \text{ with } T_\ell < |k^1|_\infty \leq T_{\ell+n}\},$$

and note that $D_n \cap I_{\ell,n} = D_{\ell,n} \cap I_{\ell,n}$.

Since $D_{\ell,n} \in \sigma((Y_k)_{|k^1|_\infty > T_\ell})$, it is independent of F , and we find

$$\mathbb{P}(D_{\ell+n}, F | I_{\ell,n}) = \mathbb{P}(D_{\ell,n}, F | I_{\ell,n}) = \mathbb{P}(D_{\ell,n} | I_{\ell,n}) \mathbb{P}(F | I_{\ell,n}) = \mathbb{P}(D_{\ell+n} | I_{\ell,n}) \mathbb{P}(F | I_{\ell,n}).$$

Conditioning on $I_{\ell,n}^C$ we note that $I_{\ell,n}^C \subseteq D_{\ell+n}^C$ and, hence,

$$\mathbb{P}(D_{\ell+n}, F | I_{\ell,n}^C) = 0 = \mathbb{P}(D_{\ell+n} | I_{\ell,n}^C) \mathbb{P}(F | I_{\ell,n}^C).$$

Similar arguments show that $D_{\ell+n}^C$ is independent of F conditional on $\sigma(I_{\ell,n})$.

Next we establish that there exists a constant $C_{d,m}$, which depends only on m , and d such that

$$\mathbb{P}(I_{\ell,n}^C) \leq C_{d,m} (T_\ell / T_{\ell+n})^m.$$

This follows from the following series of calculations that estimate $\mathbb{P}(I_{\ell,n})$ using that $(1-x) \geq e^{-2x}$ for $0 < x \leq 1/2$ and the fact that there is a dimensional constant $C_{d,m}$ such that, for all $R \geq 1$,

$$\sum_{\ell \in \mathbb{Z}^{d-m}, |\ell|_\infty \geq R} \frac{1}{|\ell|_\infty^d} \leq C_{d,m} R^{-m}.$$

We have:

$$\begin{aligned} \mathbb{P}(I_{\ell,n}) &= \prod_{|k^1|_\infty \leq T_\ell} \left(1 - \frac{1}{\max\{|k^2|_\infty, T_{\ell+n}\}^d}\right) \geq \exp \left(-2 \sum_{|k^1|_\infty \leq T_\ell} \frac{1}{\max\{|k^2|_\infty, T_{\ell+n}\}^d}\right) \\ &= \exp \left(-2T_\ell^m \sum_{z \in \mathbb{Z}^{d-m}} \frac{1}{\max\{|z|_\infty, T_{\ell+n}\}^d}\right) \\ &= \exp \left(-2T_\ell^m T_{\ell+n}^{-d} T_{\ell+n}^{d-m} - 2T_\ell^m \sum_{\ell \in \mathbb{Z}^{d-m}, |\ell|_\infty \geq T_{\ell+n}} \frac{1}{|\ell|_\infty^d}\right) \\ &\geq \exp(-2(1 + C_{d,m})(T_\ell / T_{\ell+n})^m) \geq 1 - C_{d,m} (T_\ell / T_{\ell+n})^m. \end{aligned}$$

The conditional independence for $E \in \sigma(D_1, \dots, D_\ell)$ and $F \in \sigma(D_{\ell+n})$ yields

$$\begin{aligned} \mathbb{P}(E \cap F) &= \mathbb{E}[\mathbb{E}(\mathbf{1}_E \mathbf{1}_F | \sigma(I_{\ell,n}))] = \mathbb{E}[\mathbb{E}(\mathbf{1}_E | \sigma(I_{\ell,n})) \mathbb{E}(\mathbf{1}_F | \sigma(I_{\ell,n}))] \\ &= \mathbb{E}[\mathbb{P}(E | I_{\ell,n}) \mathbb{P}(F | I_{\ell,n}) \mathbf{1}_{I_{\ell,n}} + \mathbb{P}(E | I_{\ell,n}^C) \mathbb{P}(F | I_{\ell,n}^C) \mathbf{1}_{I_{\ell,n}^C}] \\ &= \mathbb{P}(E | I_{\ell,n}) \mathbb{P}(F | I_{\ell,n}) \mathbb{P}(I_{\ell,n}) + \mathbb{P}(E | I_{\ell,n}^C) \mathbb{P}(F | I_{\ell,n}^C) \mathbb{P}(I_{\ell,n}^C) \end{aligned}$$

Using the previous decomposition and the bound on the probability of $I_{\ell,n}^C$, we find

$$|\mathbb{P}(E \cap F) - \mathbb{P}(E) \mathbb{P}(F)| \leq 2\mathbb{P}(I_{\ell,n}^C) \leq C_{d,m}(T_\ell/T_{\ell+n})^m = C_{d,m}2^{-md}.$$

□

We continue with:

Proof of Lemma 2.3. We check now, as in [18], that $\inf_{n \geq 1} \mathbb{P}(C_n) > 0$ and $\inf_{n \geq 1} \mathbb{P}(D_n) > 0$.

The lower bound for $\mathbb{P}(C_n)$ follows from

$$1 - \mathbb{P}(C_n) = (1 - \mathbb{P}(X_0 = T_n))^{\theta^d(T_n^d - T_{n-1}^d)} = (1 - \frac{\alpha_d}{T_n^d})^{\theta^d T_n^d(1-2^{-d})} \leq \exp(-\alpha_d^2 \theta^d). \quad (2.16)$$

Note that $n \geq n_\theta$, so that $\theta T_{n-1} \geq 1$, guarantees that θT_n and θT_{n-1} are both integers.

For the lower bound for $\mathbb{P}(D_n)$, we use that there exists a dimensional constant $C_{d,m}$ such that, for all $R \geq 1$, $\sum_{\ell \in \mathbb{Z}^{d-m}, |\ell|_\infty \geq R} \frac{1}{|\ell|_\infty^d} \leq C_{d,m} R^{-m}$, and we proceed with the following estimates:

$$\begin{aligned} \mathbb{P}(D_n) &= \prod_{|k^1|_\infty \leq T_n} \left(1 - \frac{1}{\max\{|k^2|_\infty, T_n\}^d}\right) \geq \exp\left(-2 \sum_{|k^1|_\infty \leq T_n} \frac{1}{\max\{|k^2|_\infty, T_n\}^d}\right) \\ &\geq \exp\left(-2T_n^m \sum_{\ell \in \mathbb{Z}^{d-m}} \frac{1}{\max\{|\ell|_\infty, T_n\}^d}\right) \\ &= \exp\left(-2T_n^m T_n^{-d} T_n^{d-m} - T_n^m \sum_{\ell \in \mathbb{Z}^{d-m}, |\ell|_\infty \geq T_n} \frac{1}{|\ell|_\infty^d}\right) \geq \exp(-2(1 + C_{d,m})). \end{aligned}$$

□

2.5. The mixing properties of the random field V . We investigate the mixing properties of V and work only with its \mathbb{Z}^d -stationary version. The reason is that the construction of the \mathbb{R}^d stationarity described earlier yields a random field which is ergodic but not mixing. We remark that one could probably maintain all the desired properties of V including both \mathbb{R}^d -stationarity and the mixing properties proven below in Lemma 2.6 by basing the definition of V on vertical and horizontal line/plane segments placed on an independent Poisson family of points rather than the \mathbb{Z}^d lattice points.

The total variation based mixing distance between two σ -algebras $\mathcal{G}_1, \mathcal{G}_2$ is defined

$$\alpha(\mathcal{G}_1, \mathcal{G}_2) := \sup\{|\mathbb{P}(E \cap F) - \mathbb{P}(E) \mathbb{P}(F)| : E \in \mathcal{G}_1 \text{ and } F \in \mathcal{G}_2\}. \quad (2.17)$$

For $A \subset \mathbb{R}^d$ a Borel set, \mathcal{F}_A is the σ -algebra generated by the random variables $(V(x))_{x \in A}$. The α -mixing coefficients associated with the random field V are given by

$$\alpha(r) := \sup\{\alpha(\mathcal{F}_A, \mathcal{F}_B) : \text{with } A, B \subset \mathbb{R}^d \text{ and } \inf_{x \in A, y \in B} |x - y| \geq r\}. \quad (2.18)$$

The following result was shown in [18] for $d = 2$.

Theorem 2.5. *The strong mixing coefficient $\alpha(r)$ associated with the random field V defined in Section 2.3 does not converge to 0 as $r \rightarrow \infty$.*

In this section we show that V actually has a mixing property which is weaker than α -mixing. Although it does not seem to be a sufficiently strong to be used to prove homogenization, we make note of it here.

The following notion of mixing was introduced in Bramson, Zeitouni and Zerner [8]. Given $\gamma > 0$, the random field V is said to be polynomially mixing with order γ , if, for every finite subset Λ of \mathbb{Z}^d with fattening $\hat{\Lambda} = \cup_{k \in \Lambda}([-1/2, 1/2]^d + k)$,

$$\sup_{\ell \in \mathbb{Z}^d \setminus \{0\}} |\ell|^\gamma \alpha(\mathcal{F}_{\hat{\Lambda}}, \mathcal{F}_{\hat{\Lambda}+\ell}) < +\infty. \quad (2.19)$$

In the next lemma we prove two mixing estimates for V . The second, which is a corollary of the first, implies that V is polynomially mixing with order $\gamma = \frac{dm}{d+m}$.

After our work was completed, we were informed that a mixing estimate similar to part (2.21) below has also been derived by Armstrong and Ziliotto [2].

Lemma 2.6. (i) *There exists $C = C(d, m) > 0$ such that, for all finite $\Lambda \subset \mathbb{Z}^d$ and $R > 1$,*

$$\alpha(\mathcal{F}_{\hat{\Lambda}}, \mathcal{F}_{(\hat{\Lambda}+Q_R)^c}) \leq C \sum_{\ell \in \Lambda} \frac{1}{(d(\ell, \hat{\Lambda}^C) + R)^{\frac{dm}{d+m}}} \leq \frac{C|\Lambda|}{R^{\frac{dm}{d+m}}}. \quad (2.20)$$

(ii) *There exists $C(d, m) > 0$ so that, for all finite $\Lambda \subset \mathbb{Z}^d$ and $\ell \in \mathbb{Z}^d$,*

$$\alpha(\mathcal{F}_{\hat{\Lambda}}, \mathcal{F}_{\hat{\Lambda}+\ell}) \leq C|\Lambda| \frac{1}{1 + (|\ell| - \text{diam}(\hat{\Lambda}))_+^{\frac{dm}{d+m}}}. \quad (2.21)$$

It will be useful in the proof to define a version of V with localized dependence on the coefficients (X_k, Y_k) . Given a subset (finite or infinite) of the lattice $\Lambda \subseteq \mathbb{Z}^d$, we define V^Λ by the same process as in subsection 2.3, except using only the (X_k, Y_k) with $k \in \Lambda$, that is giving mark $M_k = 0$ for every $k \notin \Lambda$.

Proof of Lemma 2.6. In order to keep things simple, it is convenient to view V as defined just on the integer lattice \mathbb{Z}^d .

Let $V_* : \mathbb{Z}^d \rightarrow \{-1, 0, 1\}$ to be the restriction of V to the integer lattice, and, for any set $\Lambda \subset \mathbb{Z}^d$, set

$$\mathcal{F}_*(\Lambda) := \sigma(V_*(\ell) : \ell \in \Lambda).$$

The mixing properties for V_* yield the ones of V , since, given $x \in \mathbb{R}^d$, there exists $k \in \mathbb{Z}^d$ such that $x \in Q_x := k + [0, 1)^d$, and then

$$V(x) \in \sigma(V_*(z) : z \text{ a corner of the lattice cube containing } Q_x). \quad (2.22)$$

Indeed observe that out of the two “horizontal” and two “vertical” faces of Q_x , there is at most one horizontal face H_x such that $d(x, H_x) < 1/2$ and at most one vertical face R_x such that $d(x, R_x) < 1/2$. Then

$$V(x) = 1 - 2d(x, R_x) \text{ if and only if, for all the corners } z \text{ of } R_x, V_*(z) = 1,$$

and

$$V(x) = -1 + 2d(x, H_x) \text{ if and only if, for all the corners } z \text{ of } H_x, V_*(z) = -1,$$

and, if neither of the above events hold, then $V(x) = 0$. It follows that $V(x)$ is a function of $V_*(z)$ for z which are corners of the lattice cell Q_x .

We outline next the proof of the mixing properties for V_* . For each $\ell \in \mathbb{Z}^d$ and $L \geq 5$, we construct an event $I(\ell, L)$ such that $V_*(\ell)$ conditioned upon $I(\ell, L)$ is independent of all the X_k, Y_k for $|k - \ell| \geq L$. Also for each $\ell \in \mathbb{Z}^d$, $L \geq 5$ and $R \geq L + 5$, we construct an event $J(\ell, L, R)$ such that, for every $|x - \ell| \geq R$, $V(x)$ conditioned upon $J(\ell, L, R)$ is independent of all the X_k, Y_k with $|k| < L$. Then, we choose $\lambda(R)$ to maximize the probability of the event

$$K(\ell, R) := I(\ell, (1 + \lambda)^{-1}R) \cap J(\ell, (1 + \lambda)^{-1}R, R) \text{ for } \ell \in \mathbb{Z}^d, R \geq 5.$$

Conditioning on K , an event which will have probability close to 1, we obtain the desired mixing estimates.

To simplify the notation, we assume that the events $I(\ell, L)$ and $J(\ell, L, R)$ are centered at $\ell = 0$, since the extension to $\ell \neq 0$ follows by the stationary translations. Next we write I, J to mean $I(0, L)$ and $J(0, L, R)$ respectively.

Let $L \geq 5$ and $R \geq L + 5$ and define $J(0, L, R)$ by

$$J(0, L, R) := \{\forall |\ell| < L, X_\ell \vee Y_\ell < (R - L)\}. \quad (2.23)$$

Computations similar to the ones in the previous subsection, which we do not repeat here, yield the lower bound

$$\mathbb{P}(J) \geq (1 - c_d(R - L)^{-d})^{c_d L^d} \geq 1 - \frac{c_d}{(\frac{R}{L} - 1)^d}. \quad (2.24)$$

To make precise the meaning of the conditional independence, we define the conditional probability measure $\mathbb{P}(\cdot|J)$ on \mathcal{F} by

$$\mathbb{P}(E|J) := \mathbb{P}(E \cap J) / \mathbb{P}(J) \quad \text{for } E \in \mathcal{F}.$$

We emphasize this most elementary definition of conditional probability in order to make clear the fact that we are not using the typical measure theoretic conditioning.

The claim is that, for all $E \in \mathcal{F}_*(\mathbb{Z}^d \setminus Q_R)$ and $F \in \sigma(X_k, Y_k : k \in Q_L)$,

$$\mathbb{P}(E \cap F|J) = \mathbb{P}(E|J)\mathbb{P}(F|J). \quad (2.25)$$

We also make use of the localized version of the coefficient field V_*^Λ defined for any set $\Lambda \subset \mathbb{Z}^d$, which, by its construction, is $\sigma(X_k, Y_k : k \in \Lambda)$ measurable.

The claim is that

$$V_*^{\mathbb{Z}^d \setminus Q_L}(\ell) \mathbf{1}_J = V_*(\ell) \mathbf{1}_J \quad \text{for all } \ell \in \mathbb{Z}^d \setminus Q_R. \quad (2.26)$$

The conditional independence (2.25) then follows immediately from (2.26).

If $|\ell| \geq R$, then either $|\ell^1| \geq R$ or $|\ell^2| \geq R$. Since the arguments are similar, here we assume that $|\ell^1| \geq R$. Let $H_k := k + [-X_{k/2}, X_{k/2}]^m \times \{0\}^{d-m}$ be a horizontal segment which is centered at some $k \in \mathbb{Z}^d$ and passes through ℓ . In order for H_k to be marked by a segment from Q_L , it must intersect the set $\{z : |z^1| \leq L\}$ and so it is necessary that $X_k \geq (R - L)$. On the event J , for any $k \notin Q_R$ the segment H_k cannot be marked for deletion by any of the Y_j for $j \in Q_L$ since $Y_j < X_k$. If V_k is a vertical segment passing through ℓ , then $k^1 = \ell^1$ and so $|k^1| \geq R$. For a horizontal segment centered at a point of Q_L to delete V_k it needs to have length at least $2(R - L)$ which is again not possible on the event J . It follows that $V_*(\ell) \mathbf{1}_J = V_*^{\mathbb{Z}^d \setminus Q_L}(\ell) \mathbf{1}_J$.

To define $I(0, L)$ we need the following four events:

$$I_h := \left\{ \begin{array}{l} \text{for all } |k| \geq L \text{ such that } k^2 \neq 0 \quad \text{either } Y_{(0,\ell)} < 2 \max\{|k^2 - \ell|, |\ell|\} \text{ for all } \ell \in \mathbb{Z}^{d-m}, \\ \text{or } X_k < \max\{2|k^1|, |k^2|\} \end{array} \right\}$$

$$II_h := \{X_k < 2|k^1| \text{ for all } |k| \geq L \text{ such that } k^2 = 0\}$$

$$I_v := \left\{ \begin{array}{l} \text{for all } |k| \geq L \text{ such that } k^1 \neq 0 \quad \text{either } X_{(\ell,0)} < 2 \max\{|k^1 - \ell|, |\ell|\} \text{ for all } \ell \in \mathbb{Z}^m, \\ \text{or } Y_k < \max\{2|k^2|, |k^1|\} \end{array} \right\}$$

$$II_v := \{Y_k < 2|k^2| \text{ for all } |k| \geq L \text{ such that } k^1 = 0\}.$$

Note that I_h and I_v are independent since they are defined in terms of disjoint sets of X, Y , and, moreover, $II_v \subset I_h$ and $II_h \subset I_v$.

It follows that $I := I_h \cap II_h \cap I_v \cap II_v = I_h \cap I_v$ is the intersection of two independent events. We remark that, although they appear redundant, the events II_h, II_v will have a different meaning in the proof of conditional independence.

In the sequel we make use of the estimate

$$\mathbb{P}(I) \geq 1 - CL^{-m}; \quad (2.27)$$

its proof is presented at the end of the ongoing one.

To prove conditional independence, as with the argument for J , we use the localization $V_*^{Q_L}$ which is independent of X_k, Y_k for $k \in \mathbb{Z}^d \setminus Q_L$ and aim to show that,

$$V_*(0) = V_*^{Q_L}(0) \text{ on the event } I. \quad (2.28)$$

Then, for any $E \in \mathcal{F}_*(\{0\})$, we can write $E \cap I = E' \cap I$, where E' is an event in $\sigma(V_*^{Q_L}(0))$, and, hence, $\mathbb{P}(E \cap F|I) = \mathbb{P}(E' \cap F|I)$ and $\mathbb{P}(E|I) = \mathbb{P}(E'|I)$.

Moreover, it follows from (2.27) that, for any measurable set $A \in \Omega$,

$$|\mathbb{P}(A|I) - \mathbb{P}(A)| = \mathbb{P}(A \cap I^C)/\mathbb{P}(I) \leq CL^{-m}.$$

Then, for any $F \in \sigma(X_k, Y_k : k \in \mathbb{Z}^d \setminus Q_L)$,

$$\begin{aligned} |\mathbb{P}(E \cap F) - \mathbb{P}(E)\mathbb{P}(F)| &\leq |\mathbb{P}(E \cap F|I) - \mathbb{P}(E|I)\mathbb{P}(F)| + |\mathbb{P}(E \cap F) - \mathbb{P}(E \cap F|I)| + |\mathbb{P}(E) - \mathbb{P}(E|I)|\mathbb{P}(F) \\ &\leq |\mathbb{P}(E' \cap F|I) - \mathbb{P}(E'|I)\mathbb{P}(F)| + CL^{-m} \leq CL^{-m}. \end{aligned}$$

We show next that, on the event $I_h \cap II_h$, neither can the origin be on any horizontal segment centered at some $k \in \mathbb{Z}^d \setminus Q_L$ nor can a horizontal segment centered a point $k \in \mathbb{Z}^d \setminus Q_L$ mark any vertical segment containing 0.

It follows that, on the event $I_h \cap II_h$, $V_*(0)$ is independent of the $(X_k)_{k \in \mathbb{Z}^d \setminus Q_L}$. A symmetrical argument gives that $V_*(0)$ is independent of the $(Y_k)_{k \in \mathbb{Z}^d \setminus Q_L}$ on the event $I_v \cap II_v$.

The event II_h guarantees that the horizontal segments centered at $k \in (\mathbb{Z}^d \setminus Q_L) \cap \{k^2 = 0\}$ do not contain 0. The other possibility is that a horizontal line segment centered at some $|k| \geq L$ with $k^2 \neq 0$ marks a vertical segment centered at a point $(0, \ell)$ for an $\ell \in \mathbb{Z}^{d-m}$ which crosses both $(0, k^2)$ and 0. This is not possible on the event I_h . Indeed on I_h either (i) there is no such vertical segment crossing both $(0, k^2)$ and 0, since this would require $Y_{(0, \ell)} \geq 2 \max\{|k^2 - \ell|, |\ell|\}$, or (ii) $X_k < \max\{2|k^1|, |k^2|\}$. If (ii) holds, then either $X_k < 2|k^1|$ or $X_k < |k^2|$. In the former case, the horizontal segment centered at k does not cross the vertical axis and, hence, cannot mark any point $(0, \ell)$. In the latter case, any vertical segment passing through both $(0, k^2)$ and 0 must have length at least $|k^2|$ and so it is longer than X_k and will not be marked by X_k .

Since (2.27) and (2.24) give

$$\mathbb{P}(K) \geq 1 - C((1 + \lambda)^m R^{-m} - \lambda^{-d}),$$

we choose $\lambda(R) := R^{\frac{m}{d+m}}$ to maximize the right hand side, and, hence, we find

$$\mathbb{P}(K) \geq 1 - CR^{-\frac{dm}{d+m}}.$$

Let Λ be finite a subset of \mathbb{Z}^d . Next we prove (2.20) and observe that (2.21) follows from (2.20) since, if $|\ell| \geq \text{diam}(\Lambda)$, then

$$\Lambda + \ell \subset (\Lambda + Q_R)^C \text{ with } R = |\ell| - \text{diam}(\hat{\Lambda}).$$

Let $E \in \mathcal{F}(V_*(z) : z \in \Lambda)$, $F \in \mathcal{F}(V_*(z) : z \in (\Lambda + Q_R)^C)$ and define $\hat{K} := \cap_{\ell \in \Lambda} K(\ell, d(\ell, \Lambda^c) + R)$. It follows from (2.27) and (2.24) that

$$\mathbb{P}(\hat{K}) \geq 1 - C \sum_{\ell \in \Lambda} \frac{1}{(d(\ell, \Lambda^c) + R)^{\frac{dm}{d+m}}}.$$

Then, by a calculation similar to the one in Lemma 2.4 and using that E and F are independent conditional on the event \hat{K} , we find, for some $C > 0$,

$$|\mathbb{P}(E \cap F) - \mathbb{P}(E)\mathbb{P}(F)| \leq C \sum_{\ell \in \Lambda} \frac{1}{(d(\ell, \Lambda^C) + R)^{\frac{dm}{d+m}}}.$$

We conclude with the proof of (2.27). The key step is a lower bound on the probability of I_h —note that a very similar computation works for I_v :

$$\begin{aligned} \mathbb{P}(I_h) &= \prod_{|k| \geq L, k^2 \neq 0} \mathbb{P}(\text{either } Y_{(0,\ell)} < 2 \max\{|k^2 - \ell|, |\ell|\} \text{ for all } \ell \in \mathbb{Z}^{d-m} \text{ or } X_k < \max\{2|k^1|, |k^2|\}) \\ &= \prod_{|k| \geq L, k^2 \neq 0} (1 - \mathbb{P}(\exists \ell \in \mathbb{Z}^{d-m} \text{ such that } Y_{(0,\ell)} \geq 2 \max\{|k^2 - \ell|, |\ell|\} \text{ and } X_k \geq \max\{2|k^1|, |k^2|\})) \\ &= \prod_{|k| \geq L, k^2 \neq 0} (1 - \mathbb{P}(X_k \geq \max\{2|k^1|, |k^2|\})(1 - \mathbb{P}(Y_{(0,\ell)} < 2 \max\{|k^2 - \ell|, |\ell|\} \text{ for all } \ell \in \mathbb{Z}^{d-m}))) \\ &= \prod_{|k| \geq L, k^2 \neq 0} (1 - \mathbb{P}(X_k \geq \max\{2|k^1|, |k^2|\})(1 - \prod_{\ell \in \mathbb{Z}^{d-m}} \mathbb{P}(Y_{(0,\ell)} < 2 \max\{|k^2 - \ell|, |\ell|\}))). \end{aligned}$$

Before continuing with the full computation, we estimate

$$\begin{aligned} \prod_{\ell \in \mathbb{Z}^{d-m}} \mathbb{P}(Y_{(0,\ell)} < 2 \max\{|k^2 - \ell|, |\ell|\}) &\geq \prod_{\ell \in \mathbb{Z}^{d-m}} (1 - \frac{C}{(|\ell| + |k^2 - \ell|)^d}) \geq \exp(- \sum_{\ell \in \mathbb{Z}^{d-m}} \frac{C}{(|\ell| + |k^2 - \ell|)^d}) \\ &\geq \exp(- \sum_{\ell \in \mathbb{Z}^{d-m}} \frac{C}{(|\ell| + |k^2|)^d}) \geq 1 - C|k^2|^{-m}, \end{aligned}$$

where for the third inequality we used that, since $\frac{1}{2}(|k^2| - |\ell|) \leq |k^2 - \ell|$, we have $|\ell| + |k^2 - \ell| \geq \frac{1}{2}(|k^2| + |\ell|)$.

Using this information in the estimate for $\mathbb{P}(I_h)$ we find

$$\begin{aligned} \mathbb{P}(I_h) &\geq \prod_{|k| \geq L, k^2 \neq 0} \left(1 - \mathbb{P}(X_k \geq \max\{2|k^1|, |k^2|\}) \frac{C}{|k^2|^m}\right) \geq \prod_{|k| \geq L} \left(1 - \frac{C}{(1 + |k^2|)^m(|k^1| + |k^2|)^d}\right) \\ &\geq \exp(- \sum_{|k| \geq L, k \in \mathbb{Z}^d} \frac{C}{(1 + |k^2|)^m(|k^1| + |k^2|)^d}) \geq 1 - CL^{-m}; \end{aligned}$$

in the last line we used that

$$\begin{aligned} \int_{\mathbb{R}^d \setminus B(0,L)} \frac{1}{(1 + |x^2|)^m(|x^1| + |x^2|)^d} dx &\lesssim \int_{\mathbb{R}^d} \frac{1}{(1 + |x^2|)^m(L + |x^1| + |x^2|)^d} dx \\ &= \int_{\mathbb{R}^{d-m}} \frac{1}{(1 + |x^2|)^m} \int_{\mathbb{R}^m} \frac{1}{(L + |x^2| + |x^1|)^d} dx^1 dx^2 \\ &\lesssim \int_{\mathbb{R}^{d-m}} \frac{1}{(1 + |x^2|)^m} \frac{1}{(L + |x^2|)^{d-m}} dx^2 \lesssim L^{-m}. \end{aligned}$$

A similar computation for $\mathbb{P}(I_v)$ yields

$$\mathbb{P}(I_v) \geq 1 - CL^{-(d-m)}.$$

Finally, using that $m \leq d - m$ by assumption, we get

$$\mathbb{P}(I) = \mathbb{P}(I_h \cap I_v) = \mathbb{P}(I_h)\mathbb{P}(I_v) \geq (1 - CL^{-m})(1 - CL^{-(d-m)}) \geq 1 - CL^{-m}.$$

□

2.6. Space-time random fields. We discuss here a simple consequence of Proposition 2.1. If the Hamiltonian $h \in C^2(\mathbb{R}^d)$ has super-linear growth at infinity and there exists ξ_0 such that $D^2h(\xi_0)$ is non-degenerate and has both positive and negative eigenvalues, then there exists a space-time stationary ergodic $V = V(x, t)$ for which homogenization does not take place for the Hamiltonian

$$H(\xi, x, t) = h(\xi) - V(x, t);$$

that is, if u is the solution of

$$u_t + H(Du, x, t) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad u(x, 0) = \xi_0 \cdot x,$$

then

$$\liminf_{t \rightarrow \infty} \frac{u(0, t)}{t} < \limsup_{t \rightarrow \infty} \frac{u(0, t)}{t} \quad \text{almost surely.}$$

In what follows we work for simplicity in $d = 2$. The assumption on h yields the existence of $\xi_0 \in \mathbb{R}^2$ such that the eigenvalues of $D^2h(\xi_0)$ have opposite sign. Let $p_0 := Dh(\xi_0)$. Although h does not have a strict saddle point at ξ_0 , since there is no reason that $p_0 = 0$, the Hamiltonian

$$g(\xi) = h(\xi) - p_0 \cdot \xi \quad \text{has a strict saddle-point at } \xi_0$$

and is still coercive; note that this the reason we assumed that h has superlinear growth. It then follows from Lemma 2.1 that there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a group of measure preserving transformations $(\tau_x)_{x \in \mathbb{R}^d}$ acting on $(\Omega, \mathcal{F}, \mathbb{P})$ which is ergodic, and a stationary random field $V_0 : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$ (we hide the dependence on ω as usual) so that the solution v of,

$$v_t + g(Dv) - V_0(x) = 0 \quad \text{in } \mathbb{R}^2 \times (0, \infty) \quad v(x, 0) = \xi_0 \cdot x,$$

satisfies

$$\liminf_{t \rightarrow \infty} \frac{v(0, t)}{t} < \limsup_{t \rightarrow \infty} \frac{v(0, t)}{t} \quad \text{almost surely in } \mathbb{P}.$$

Now observe that $u(x, t) := v(x - p_0 t, t)$, solves

$$u_t + g(Du) + p_0 \cdot Du - V_0(x - p_0 t) = 0 \quad \text{in } \mathbb{R}^2 \times (0, \infty) \quad u(x, 0) = \xi_0 \cdot x.$$

We need to justify that $V(x, t) := V_0(x - p_0 t)$ is indeed space-time stationary and ergodic. For this we extend the measure preserving system $(\Omega, \mathcal{F}, \mathbb{P}, (\tau_x)_{x \in \mathbb{R}^d})$ by defining $\sigma_{(x, t)} := \tau_{x - p_0 t}$. It is easily checked that $\mathbb{R}^d \times \mathbb{R}$ acts on $(\Omega, \mathcal{F}, \mathbb{P})$ in a measure preserving way via the transformations $\sigma_{(x, t)}$. Furthermore the group of transformations remains ergodic. Indeed, if $E \in \mathcal{F}$ is invariant under all the translations $\sigma_{(x, t)}$, then, in particular, it is invariant under all $\sigma_{(x, 0)} = \tau_x$ and thus, by the ergodicity of the original system, $\mathbb{P}(E) \in \{0, 1\}$. Finally we just need to check that $V(x, t, \omega)$ is stationary, a fact that follows from the identities

$$V(x, t, \omega) = V_0(x - p_0 t, \omega) = V_0(0, \tau_{x - p_0 t} \omega) = V(0, 0, \sigma_{(x, t)} \omega).$$

The properties of v imply

$$\liminf_{t \rightarrow \infty} \frac{u(p_0 t, t)}{t} < \limsup_{t \rightarrow \infty} \frac{u(p_0 t, t)}{t} \quad \text{almost surely in } \mathbb{P}. \quad (2.29)$$

Although it is not so obvious, (2.29) is equivalent to the analogous statement at $u(0, t)$. This is a consequence of a well known fact, usually stated for the approximate corrector problem without time dependence; see [5] Lemma 5.1.

Define

$$H_*(\xi_0) := \liminf_{t \rightarrow \infty} \frac{u(0, t)}{t} \quad \text{and} \quad H^*(\xi_0) = \limsup_{t \rightarrow \infty} \frac{u(0, t)}{t}$$

The uniform Lipschitz continuity of u yields that H_* and H^* are translation invariant and therefore, by ergodicity, are almost surely constant.

It is now immediate consequence of (??) and (2.29) that $H_*(\xi_0) < H^*(\xi_0)$.

3. QUANTITATIVE HOMOGENIZATION FOR HAMILTONIANS WITH STAR-SHAPED SUB-LEVEL SETS

We identify a new class of nonconvex Hamiltonians, namely H 's with “quantitatively” star-shaped sub-levels, such that the Hamilton-Jacobi equation (1.1) homogenizes in random media with finite range dependence.

Our arguments extend to study the limit, as $\varepsilon \rightarrow 0$, of the solutions to the “viscous” Hamilton-Jacobi equation,

$$u_t^\varepsilon - \varepsilon \Delta u + H(Du^\varepsilon, \frac{x}{\varepsilon}) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \quad u(x, 0) = u_0(x), \quad (3.1)$$

always in a medium with finite range dependence, with Hamiltonians satisfying a crossover of quantitative star-shapedness and positive homogeneity. The analysis applies even to more complicated degenerate quasilinear versions of (3.1). We do not state, however, any specific results.

The goal of this section is to point out that the strong star-shapedness assumption (1.5) or, more precisely, a quantification of it (see (3.5) below), and not homogeneity is in fact the key assumption which allows the arguments of [1] to be carried out for first-order problems.

As already discussed in the introduction, the analysis relies heavily on the arguments in [1]. Instead of rewriting large portions of [1], here we point out the critical places where the star-shaped sub-level set property needs to be used and, in these cases, we present full proofs.

3.1. Outline. Before getting in the technical details we give a broad outline of the proof of quantitative homogenization from [1], which had its origin in the article [3].

As was previously established in [4, 5], an efficient way to study the homogenization of (1.1) is to look at the asymptotic behavior of the solutions to the so-called “metric problem” to a closed set $S \subset \mathbb{R}^d$. The latter is to seek, for each $\mu > 0$, a positive solution m_μ to

$$H(Dm_\mu, x) = \mu \quad \text{in } \mathbb{R}^d \setminus S \quad \text{with } m_\mu(x) \leq 0 \text{ on } S. \quad (3.2)$$

In order to make the connection between the metric and the original homogenization problems more transparent, we consider here an eikonal or level-set evolution type Hamiltonian of the form

$$H(\xi, x) = a(\frac{\xi}{|\xi|}, x)|\xi|.$$

This choice of H may seem to be restrictive, but, in fact, the metric problems for star-shaped sub-level set Hamiltonians are equivalent to metric problems for level-set evolutions; see subsection 3.3 for more details.

In view of the positive homogeneity of the Hamiltonian we are using here, it turns out that it is enough to consider the $\mu = 1$ metric problem, and to simplify the presentation, next we write m instead of m_1 .

An intuitive way to understand the role of m , is to think of $m(x)$ as being exactly the arrival time at the location x of a front starting at S and moving with outward normal velocity $a(n, \cdot)$. This can be seen by looking at $u(x, t) = m(x) - t$ which evidently solves

$$u_t + H(Du, x) = 0 \quad \text{with } \partial\{u(x, 0) = 0\} = \partial S.$$

Here we are, in particular, interested in the planar metric problem, that is (3.2) with a half plane target set S , that is

$$S = \mathcal{H}_e^\pm := \{x \in \mathbb{R}^d : \pm x \cdot e \geq 0\} \quad \text{for some } e \in S^{d-1}.$$

In this case the t -level-set of the metric problem solution m is the location at time t of a front starting from the hyperplane $\{y \in \mathbb{R}^d : y \cdot e = 0\}$ and moving with normal velocity $a(n, \cdot)$.

For the purposes of homogenization it is important to understand the long time behavior of this front. In particular, if we can find a non-random asymptotic speed $\bar{a}(e)$ for every direction e , then the homogenized Hamiltonian is $\bar{H}(\xi) = \bar{a}(\frac{\xi}{|\xi|})|\xi|$.

The uniqueness property for the metric problem, which is guaranteed by the star-shapedness condition, comes into the proof of quantitative homogenization in two important places which we discuss next in this hand waving outline.

The nice feature of geometric (level-set) evolutions is that they inherently localize the spatial dependence of the solution on the coefficients of the problem. This is an extremely useful feature in quantitative random homogenization, where one is always studying this question of how the solution depends on the coefficients of the equation.

The localization property of the geometric evolutions, stated informally, is that the location of the front at time t depends only on the region traced out by the front on the interval $(0, t)$ or in other words,

$$m(x) \text{ depends only on the values of } H \text{ for } x \in \{y : m(y) \leq m(x)\}.$$

This is an immediate consequence of the uniqueness of the level-set evolution; see Lemma 3.5, and a critical place where the comparison principle for the metric problem comes into the proof of quantitative homogenization.

The property of the previous paragraph, which is referred to as localization in sub-level sets, is the key tool in obtaining a martingale decomposition of $m(x)$ – this idea goes back to the work of Kesten [12] on first passage percolation.

Loosely speaking, let \mathcal{F}_t be the minimal filtration of the probability space which makes the set valued random variables $\{x : m(x) \leq t\}$ to be \mathcal{F}_t -measurable for all $t > 0$. The existence of \mathcal{F}_t is, of course, a bit tricky are considering continuous and not discrete in time setting.

The martingale we are interested in is

$$\mathbb{E}[m(x)|\mathcal{F}_t] - \mathbb{E}[m(x)].$$

The lower bound on the front propagation speed implied by the coercivity yields that, if $t \gtrsim x \cdot e$, then almost surely $x \in \{y \in \mathbb{R}^d : m(y) \leq t\}$. It then follows from the localization in sub-level sets property that $m(x) \in \mathcal{F}_t$ and, hence,

$$m(x) - \mathbb{E}[m(x)] = \mathbb{E}[m(x)|\mathcal{F}_t] - \mathbb{E}[m(x)] = \sum_{k=0}^{[t]} \mathbb{E}[m(x)|\mathcal{F}_{k+1}] - \mathbb{E}[m(x)|\mathcal{F}_k],$$

which is a sum of (bounded) martingale differences.

Then a classical concentration estimate, known as Azuma's inequality, gives the following estimate on the probability that $m(x)$ deviates too much from its mean $\mathbb{E}m(x)$:

$$\mathbb{P}(|m(x) - \mathbb{E}m(x)| \geq \lambda(x \cdot e)^{1/2}) \leq C \exp(-c\lambda^2). \quad (3.3)$$

Having established an estimate on the concentration of $m(te)$ about its mean $\mathbb{E}m(te)$, we are left with the task to understand the limiting behavior of $\mathbb{E}m(te)$ as $x \cdot e \rightarrow +\infty$.

It turns out (see Proposition 3.6) that $\mathbb{E}m(te)$ is approximately linear in t as $t \rightarrow \infty$, a fact that is expressed in the estimate

$$|\mathbb{E}m((t+s)e) - \mathbb{E}m(te) - \mathbb{E}m(se)| \leq Ct^{1/2} \log^{1/2}(1+t) \quad \text{for } 1 \leq s \leq t.$$

This approximate linearity comes from the concentration estimate combined with a semi-group (uniqueness again) property of m . The idea of the proof is as follows. The concentration about the mean established in (3.3) implies that, when λ is large, $|m(te) - \mathbb{E}m(te)| \leq \lambda t^{1/2}$ with high probability. In fact, using the stationarity, continuity and a union bound, is not hard to show that, except for an event of very small probability, $m(x+te) - \mathbb{E}m(x+te)$ is not too much larger than $t^{1/2}$ for any $x \in B_{Rt}(0) \cap \{y \in \mathbb{R}^d : y \cdot e = 0\}$ for large R . In other words, m looks, with reference to the scale t , like the constant $\mathbb{E}m(te)$ on $(\partial\mathcal{H}_e^+ + te) \cap B_{Rt}(te)$. Here is where the semi-group/quantitative uniqueness property comes in again, since we can now compare m with the metric problem solution in $\mathcal{H}_e^+ + te$ with boundary data $\mathbb{E}m(te)$ on $\partial(\mathcal{H}_e^+ + te)$. Then the fluctuations estimate give that, again with high

probability, m will be close to $\mathbb{E}m(se) + \mathbb{E}m(te)$ at $(t+s)e$. This is the second important way that uniqueness comes into the proof of quantitative homogenization.

Although we have been trying to emphasize in this outline the important role played by the uniqueness property of the metric problem, we conclude our summary by explaining why uniqueness could perhaps be not as important as it seems. If there are indeed multiple (relevant) solutions of the metric problem, it is conceivably possible that there could be some way of keeping track of each solution separately, for example, by a new metric type problem now with uniqueness. In other words given a particular solution of the metric problem one could attempt to find, varying μ , a unique continuation of that branch of solutions. In a sense this is the approach carried out by [6] for a particular class of equations.

3.2. The assumptions on the Hamiltonian and the random field and some extensions. Aiming to avoid unnecessary complications we make the following two assumptions, the first being a normalization and the second a simplification:

$$\text{ess sup } H(0, 0) = 0 \quad \text{and} \quad \mathbb{P}(H(0, 0) = 0) = c_0 > 0. \quad (3.4)$$

Given the normalization $\text{ess sup } H(0, x) = 0$, it is shown in [3] that $\mathbb{P}(H(0, 0) = 0)$ controls the lower deviations of the approximate correctors $\mathbb{P}(-\delta v^\delta(0, \xi) \ll -\delta)$ for all ξ . As a consequence, the homogenized Hamiltonian, if it exists, satisfies $\min \bar{H} \geq 0$. This result depends only on coercivity and no additional structural properties of the Hamiltonian.

Next we make precise our assumption on the quantitative star-shapedness of the $\mu > 0$ sub-level sets $\{\xi : H(\xi, x) \leq \mu\}$. We assume that there exists a modulus $\omega : [0, \infty) \rightarrow [0, \infty)$, which is positive for $\mu > 0$ such that, for all $(\xi, x) \in \mathbb{R}^d \setminus \{0\} \times \mathbb{R}^d$,

$$\frac{\xi}{|\xi|} \cdot D_\xi H(\xi, x) > \omega(H(\xi, x) \vee 0). \quad (3.5)$$

Note that (3.5) indeed implies that the μ positive sub-level sets of H are strictly star-shaped.

We also need the standard coercivity, growth and continuity bounds on the Hamiltonian, that is we assume that there are exist $0 < c_0, C_0 < \infty$ and $p \geq q \geq 1$ such that, for all $\xi, x \in \mathbb{R}^d$,

$$c_0|\xi|^q - C_0 \leq H(\xi, x) \leq C_0|\xi|^p + C_0, \quad (3.6)$$

and

$$|D_x H(\xi, x)| + (1 \vee |\xi|)|D_\xi H(\xi, x)| \leq C_0(1 \vee |\xi|^p). \quad (3.7)$$

When working with (1.1), to simplify statements and shorten writing we combine the above assumption on H in

$$H \text{ satisfies (3.5), (3.6), (3.7) and (3.4) and we call } \text{data} = (d, c_0, C_0, q, p). \quad (3.8)$$

We work on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here Ω is the set of all fields of Hamiltonians satisfying the above assumptions, that is

$$\Omega := \{H : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} : H \text{ satisfies (3.8)}\}. \quad (3.9)$$

For each $U \subseteq \mathbb{R}^d$ Borel we define the cylinder σ -algebra $\mathcal{F}(U)$

$$\mathcal{F}(U) := \sigma(H \mapsto H(\xi, x) : x \in U, \xi \in \mathbb{R}^d).$$

Then the σ -algebra \mathcal{F} is taken to be the largest of the $\mathcal{F}(U)$,

$$\mathcal{F} := \mathcal{F}(\mathbb{R}^d). \quad (3.10)$$

The set Ω is endowed with the group of translations $(T_y)_{y \in \mathbb{R}^d}$ given, for each $y \in \mathbb{R}^d$, by

$$T_y H(\cdot, \cdot) := H(\cdot, \cdot + y).$$

We assume that the probability measure \mathbb{P} , which remains fixed throughout this section, is stationary and has unit range of dependence, that is, respectively,

$$\text{the map } T_y \text{ is } \mathbb{P}\text{-preserving for every } y \in \mathbb{R}^d, \quad (3.11)$$

and

$$\begin{cases} \text{the } \sigma\text{-algebras } \mathcal{F}(U) \text{ and } \mathcal{F}(V) \text{ are} \\ \mathbb{P}\text{-independent for every pair of Borel sets } U, V \text{ in } \mathbb{R}^d, \text{ with } d(U, V) \geq 1. \end{cases} \quad (3.12)$$

We summarize the properties in

$$H \mapsto H(\cdot, x) \text{ is a stationary and 1-dependent random field of Hamiltonians on } \mathbb{R}^d. \quad (3.13)$$

We discuss briefly more general types of Hamiltonians where these methods could be applied. The Hamiltonian

$$H(\xi, x) = |\xi|^2 - b(x) \cdot \xi,$$

has star-shaped level sets of H in a uniform way, but the center of the star-shapedness is varying in x . Indeed,

$$H(\xi, x) = |\xi - b(x)|^2 - \frac{1}{4}|b(x)|^2,$$

and it is clear that the sub-level sets of $H(\cdot, x)$ are star-shaped with respect to $b(x)$.

If $b(x) = DB(x)$, where B is a bounded C^1 -vector field, the transformation to,

$$v(x) = u(x) - B(x),$$

yields that, if u solves $H(Du, x) = \mu$ in $U \subset \mathbb{R}^d$, then v solves

$$|Dv|^2 - \frac{1}{4}|b(x)|^2 = \mu,$$

which is an equation satisfying the assumptions of the previous section.

The observation above shows that we can work with Hamiltonians H , such that, that there exists a bounded stationary random potential field $B \in C^1(\mathbb{R}^d; \mathbb{R}^d)$ with finite range of dependence such that (3.8) is satisfied by the Hamiltonian

$$G(Du, x) := H(Du + DB, x).$$

In a different direction, the individual sub-level sets of $H(\cdot, x)$ for fixed x can be uniformly star-shaped with respect to different points. Indeed we can assume that, for every $\mu > 0$, there exists ξ_μ such that, for every $x \in \mathbb{R}^d$, $\{\xi : H(\xi, x) \leq \mu\}$ is star-shaped with respect to ξ_μ and

$$\frac{\xi - \xi_\mu}{|\xi - \xi_\mu|} \cdot D_\xi H(\xi, x) \geq \omega(\mu) \text{ for all } x \in \mathbb{R}^d, \xi \in \{H(\xi, x) = \mu\}. \quad (3.14)$$

Quite similar estimates to those we derive below will hold in this setting as well.

Several other generalizations are evident, including a combination of the two we have mentioned.

Another possibility, out of reach of our current methods, is to consider Hamiltonians with the property that every connected component of every level set of H is the boundary of a star-shaped set. Formally one can separate out each connected component of each level set getting an eikonal type equation for each, but the connection with the original equation Lemma 3.1 is lost.

We conclude the discussion about the general assumptions commenting that the main property needed to study the “viscous” Hamilton-Jacobi problem (3.1) turns out to be something between star-shaped sub-levels and homogeneity.

Indeed the conditions is that, for every $\mu > 0$, there exists $\xi_\mu \in \mathbb{R}^d$ such that, for every $x \in \mathbb{R}^d$, the level set $\{\xi \in \mathbb{R}^d : H(\xi, x) \leq \mu\}$ is star-shaped with respect to ξ_μ and,

$$\frac{\xi - \xi_\mu}{|\xi - \xi_\mu|} \cdot D_\xi H(\xi, x) \geq \mu \vee 0 \text{ for all } x \in \mathbb{R}^d \text{ and } \xi \in \{\xi' \in \mathbb{R}^d : H(\xi', x) = \mu\}. \quad (3.15)$$

Hamiltonians which are p -homogeneous H with $p \geq 1$ satisfy (3.15). However, (3.15) allows for more general H 's like, for example, logarithmic terms $H(\xi) = |\xi| \log(1 + |\xi|)$ or direction dependent homogeneity as, for example, $H(\xi) = |\xi|^{p(\frac{\xi}{|\xi|})}$ with $p : S^{d-1} \rightarrow [1, \infty)$.

3.3. Reduction to an eikonal/level-set equation and the radius function. In subsection 3.1 we explained in a heuristic way the homogenization proof for level-set type Hamiltonians. Here we show that metric-type problems for sub-level star-shaped Hamiltonians can be transformed into metric-type problems for a 1-homogeneous level-set type Hamiltonian. As a result the methods explained in Section 3.1 can also be applied to the class of sub-level star-shaped Hamiltonians.

Assume that (1.5) holds, that is, for every $\mu > 0$, the sub-level $\{H(\xi, x) \leq \mu\}$ is strictly star-shaped with respect to 0. Then there exists a “radius”-function $r_\mu : S^{d-1} \times \mathbb{R}^d \rightarrow (0, \infty)$ such that,

$$H(r\hat{\xi}, x) < \mu \text{ for } r < r_\mu(\hat{\xi}) \text{ and } H(r\hat{\xi}, x) > \mu \text{ for } r > r_\mu(\hat{\xi}). \quad (3.16)$$

It follows that the metric problem for H is equivalent to the one for $\tilde{H}(\xi, x) := r_\mu(\hat{\xi}, x)^{-1}|\xi|$.

Lemma 3.1. *Let U be a domain of \mathbb{R}^d and assume that (1.5) holds. Then*

$$r_\mu\left(\frac{Du}{|Du|}, x\right)^{-1}|Du| = 1 \text{ in } U \text{ if and only if } H(Du, x) = \mu \text{ in } U.$$

The proof is an almost immediate consequence of the definition of viscosity solutions.

Proof. If for some $x \in U$, $\xi \in D^+u(x, t)$, the sub-solution property, yields

$$r_\mu(\hat{\xi}, x)^{-1}|\xi| \leq 1,$$

and the definition of r_μ in (3.16) gives,

$$H(\xi, x) \leq \mu.$$

that is u is a sub-solution of $H(Du, x) \leq \mu$ in U . A similar argument works for the super-solution property as well as the other direction in the claim, \square

Given (3.8), we show that the radius function r_μ satisfies all the properties needed for homogenizing the level-set equation $|Dm_\mu| = r_\mu\left(\frac{Dm_\mu}{|Dm_\mu|}, x\right)$. These are upper and lower bounds, continuity in e , and monotonicity and continuity in μ .

Lemma 3.2. (i) *There exist constants $C, c > 0$ such that, for all $e \in S^{d-1}$ and $x \in \mathbb{R}^d$,*

$$0 < a_\mu < \mu(e, x) < A_\mu,$$

with $a_\mu = c(\mu^{1/p} \wedge \mu)$ and $A_\mu = C(\mu + C)^{1/q}$.

(ii) *For every $\mu > 0$, $e \in S^{d-1}$ and $x \in \mathbb{R}^d$,*

$$C_0^{-1}A_\mu^{1-p} \leq \frac{d}{d\mu}r_\mu(e, x) \leq \frac{1}{\omega(\mu)}.$$

Furthermore $e \rightarrow r_\mu(e, x)$ is Lipschitz continuous on S^{d-1} with Lipschitz constant $L_\mu := C_0a_\mu^{-1}\omega(\mu)^{-1}A_\mu^p$.

Proof. It follows from (3.6) that

$$\mu = H(r_\mu(e, x)e, x) \geq c_0r_\mu(e, x)^q - C_0,$$

while (3.6), (3.7) and (3.4) give

$$\mu = H(r_\mu(e, x)e, x) \leq C_0(r_\mu(e, x) + p^{-1}r_\mu(e, x)^p).$$

Then

$$c(\mu^{1/p} \wedge \mu) \leq r_\mu(e, x) \leq C(\mu + C)^{1/q}$$

and the claim follows with $A_\mu = C(\mu + C)^{1/q}$ and $a_\mu = c(\mu^{1/p} \wedge \mu)$.

Next we examine the dependence of r_μ on μ . The Lipschitz estimate in (3.6) gives

$$\mu = H(r_\mu(e, x)e, x) \leq H(r_\nu(e, x)e, x) + C_0A_\mu^{p-1}(r_\mu(e, x) - r_\nu(e, x)), \quad (3.17)$$

and, using (3.5), for any $\mu > \nu > 0$, we obtain

$$\mu = H(r_\mu(e, x)e, x) \geq H(r_\nu(e, x)e, x) + \omega(\mu) \wedge \omega(\nu)(r_\mu(e, x) - r_\nu(e, x)).$$

Rearranging the above inequalities, using that $H(r_\nu(e, x)e, x) = \nu$ and combining with (3.17) yields the continuity estimate

$$C_0^{-1}A_\mu^{1-p}(\mu - \nu) \leq (r_\mu(e, x) - r_\nu(e, x)) \leq \frac{1}{\omega(\mu) \wedge \omega(\nu)}(\mu - \nu). \quad (3.18)$$

Sending $\nu \rightarrow \mu$ and/or $\mu \rightarrow \nu$ gives the result.

Finally, we consider the Lipschitz estimate of $r_\mu(\cdot, x)$ on the unit sphere. Fix $e \in S^{d-1}$ and $\delta > 0$, let $e' \in S^{d-1}$ be such that $|e' - e| = \delta$, define λ by solving the relation

$$\lambda = 1 + 2C_0\omega(\mu)^{-1}\lambda^{p-1}r_\mu(e, x)^{p-1}\delta,$$

and note that $\lambda \rightarrow 1$ as $\delta \rightarrow 0$. In particular we can take δ sufficiently small so that $2 > \lambda > 1$.

The continuity of ω and H yield that, for δ small and $\lambda \approx 1$, $\omega(H(\lambda r_\mu(e, x)e, x)) \geq \frac{1}{2}\omega(\mu)$ and, hence, in view of (3.5),

$$H(\lambda r_\mu(e, x)e, x) \geq H(r_\mu(e, x)e, x) + (\lambda - 1)\omega(H(\lambda r_\mu(e, x)e, x)) \wedge \omega(\mu) = \mu + (\lambda - 1)\frac{1}{2}\omega(\mu),$$

and, in view of (3.7),

$$H(\lambda r_\mu(e, x)e', x) \geq \mu + (\lambda - 1)\frac{1}{2}\omega(\mu) - C_0(\lambda r_\mu(e, x))^{p-1}\delta.$$

By the choice of λ , we have $H(\lambda r_\mu(e, x)e', x) \geq \mu$ which, by the definition of $r_\mu(e', x)$, gives

$$r_\mu(e', x) \leq \lambda r_\mu(e, x) \leq r_\mu(e, x) + 2\omega(\mu)^{-1}\lambda^{p-1}C_0r_\mu(e, x)^p\delta.$$

Sending $e' \rightarrow e$ so that $\delta = |e - e'| \rightarrow 0$ and $\lambda \rightarrow 1$, we obtain the desired result

$$|D_e r_\mu(e, x)| \leq 2C_0\omega(\mu)^{-1}r_\mu(e, x)^p.$$

□

3.4. The planar metric problem. We establish some basic properties of the metric or minimal time problem for star-shaped sub-level Hamiltonians. Given a closed set $S \subset \mathbb{R}^d$ and $\mu > 0$ we look for a non-negative solution $m_\mu(x, S, H)$ to the problem,

$$H(Dm_\mu, x) = \mu \text{ in } \mathbb{R}^d \setminus S \text{ and } m_\mu = 0 \text{ on } \partial S. \quad (3.19)$$

As described in subsection 3.1, of particular interest is the planar metric problem, that is (3.19) with a half-space target set $S = \mathcal{H}_e^\pm := \{x \in \mathbb{R}^d : \pm x \cdot e \geq 0\}$, for some unit direction $e \in S^{d-1}$. A possible solution to (3.19) is the maximal sub-solution. There may exist, however, more solutions – for example this is the case when H is spatially homogeneous and has a sub-level which is not star-shaped.

We show that, for star-shaped sub-level Hamiltonians, the planar metric problem (3.19) has a comparison principle and, hence, a unique solution. This is the key reason that allows the use of the planar metric problems to prove homogenization.

In the first-order case we are considering here, we can use the transformation of subsection 3.3 to show uniqueness. In fact, uniqueness is a corollary of the uniqueness of metric problems for homogeneous Hamiltonians established, for example, in [1]. Indeed, Lemma 3.1 says that any solution to (3.19) by Lemma 3.1 is also a solution for the Hamiltonian $r_\mu^{-1}(\hat{\xi}, x)|\xi|$. Since comparison/uniqueness holds for the latter problem, the Hamiltonian being positively homogeneous, comparison/uniqueness holds for H as well. This is in essence the proof of the next lemma which is omitted.

Lemma 3.3. *Assume (3.8) and let m^1 and m^2 be respectively a subsolution and a non-negative supersolution of (3.19). Then $m^1 \leq m^2$ in $\mathbb{R}^d \setminus S$.*

3.5. The fluctuations estimate. We consider the stochastic part of the error estimate for the metric problem solutions, that is the fluctuations around the mean

$$m_\mu(x, S) - \mathbb{E}m_\mu(x, S).$$

We state without details an estimate of the stochastic fluctuations for $S \subseteq \mathbb{R}^d$ compact. The claim can be derived as a corollary of Proposition 3.1 of [1] using the transformation presented previously. We remark that the dependence of the error on μ is better than the one obtained in [1] because here we consider only the first-order problem. The calculation of the exact constant is tedious and requires a careful reading through the proof of [1].

Proposition 3.4 (Fluctuations estimate). *Assume (3.8) and (3.13) and let $S \subseteq \mathbb{R}^d$ be a compact set. Then, for every $\bar{\mu} > 1$, there exists $C = C(d, \bar{\mu}) \geq 1$ such that, for all $\mu \in (0, \bar{\mu}]$ and $x \in \mathbb{R}^d \setminus S$,*

$$\mathbb{P}(|m_\mu(x, S) - \mathbb{E}[m_\mu(x, S)]| \geq \lambda) \leq C \exp\left(-\frac{a_\mu \lambda^2}{C(1 + d(x, S))}\right). \quad (3.20)$$

As described in subsection 3.1, Proposition 3.4 is proved by a decomposition of the difference $m_\mu(x, S) - \mathbb{E}[m_\mu(x, S)]$ as a sum of bounded martingale differences. The key lemma (Lemma 3.1) needed for this martingale decomposition is the localization in sub-level sets property of geometric evolutions. Informally speaking, this says that $m_\mu(\cdot, S, H)$ in the sub-level $\{m_\mu \leq t\}$ depends only on the values of H there. Its proof is based on the comparison principle for the metric problem and, hence, it relies on the star-shapedness of the sub-levels.

Lemma 3.5. *Assume (3.8), (3.13) and fix $H_1, H_2 \in \Omega$ and $t > 0$. If $H_1 \equiv H_2$ in $\mathbb{R}^d \times \{x \in \mathbb{R}^d \setminus S : m_\mu(x, S, H_1) \leq t\}$, then*

$$m_\mu(\cdot, S, H_1) \leq m_\mu(\cdot, S, H_2) \text{ in } \{x \in \mathbb{R}^d \setminus S : m_\mu(x, S, H_1) \leq t\}.$$

Proof. Let $m_i := m_\mu(\cdot, S, H_i)$ with $i = 1, 2$, set $R_t := \{x \in \mathbb{R}^d \setminus S : m_\mu(x, S, H_1) \leq t\}$. fix a smooth $\phi : [0, \infty) \rightarrow [0, t]$ such that $\phi' \leq 1$ and $\phi(s) = t$ for all $s \geq t$.

It follows that $w := \phi(m_1)$ is a subsolution of $H_2(Dw, x) \leq \mu$ in $\mathbb{R}^d \setminus S$. Indeed this is the case in $\mathbb{R}^d \setminus R_t$ where w is constant there and the assumptions on H yield that constant functions are subsolutions $H(0, x) \leq 0 \leq \mu$ of $H \in \Omega$. Using that $H_1 \equiv H_2$ in R_t , $\phi' \leq 1$ and the star-shapedness condition (3.5), we find

$$H_2(Dw, x) = H_2(\phi'(m_1(x))Dm_1, x) = H_1(\phi'(m_1(x))Dm_1, x) \leq \mu \text{ in } R_t.$$

Now the comparison principle for the planar metric problem associated with the operator H_2 that is Lemma 3.3 implies that $w \leq m_2$. Finally, letting $\phi(s)$ to converge to $s \wedge t$, and, hence, $w \rightarrow m_1 \wedge t$ yields that $m_1 \leq m_2$ in the set R_t . \square

3.6. The bias estimate. Next we discuss the deterministic part of the error estimate, the convergence of the expected values $\mathbb{E}[m_\mu(x, \mathcal{H}_e^-)]$. Again our result is a corollary of [1] by the level-set transformation.

Proposition 3.6. *Let $\bar{\mu} \geq 1$, $\mu \in (0, \bar{\mu}]$ and $e \in S^{d-1}$. There exists $\bar{r}_\mu(e) > 0$ and $C = C(\text{data}, \bar{\mu}) \geq 1$ such that, for every $x \in \mathcal{H}_e^+$,*

$$|\mathbb{E}[m_\mu(x, \mathcal{H}_e^-)] - \bar{r}_\mu(e)(x \cdot e)| \leq C_\mu(x \cdot e)^{1/2} \log^{1/2}(1 + (x \cdot e)),$$

where $C_\mu = C(\text{data}, \bar{\mu})a_\mu^{-1/2}(1 + |\log L_\mu|)^{1/2}$. Moreover, $(\mu, e) \rightarrow \bar{r}_\mu(e)$ is continuous on $(0, \infty) \times S^{d-1}$.

The proof of Proposition 3.6 follows, as in [1], by a series of lemmata that are stated below without a proof. The comparison principle for the metric problem again plays a key role.

The objective is to show that $\mathbb{E}m_\mu(te, \mathcal{H}_e^-)$ is almost additive in xt .

Lemma 3.7. *For all $\mu \in (0, \bar{\mu}]$ and $t \geq s \geq 1$,*

$$|\mathbb{E}m_\mu((t+s)e, \mathcal{H}_e^-) - \mathbb{E}m_\mu(te, \mathcal{H}_e^-) - \mathbb{E}m_\mu(se, \mathcal{H}_e^-)| \leq C_\mu(s \vee t)^{1/2} \log^{1/2}(1 + s \vee t),$$

where $C_\mu = C(\bar{\mu})a_\mu^{-1/2}(1 + |\log L_\mu|)^{1/2}$.

The fluctuation estimate guarantees that, with high probability, m_μ is close to its mean in a large ball $B_{Rt}(te)$. Define

$$N_R^+(t) := \sup_{x \in B_{Rt}(te) \cap (\partial \mathcal{H}_e^+ + te)} m_\mu(x, \mathcal{H}_e^-) \quad \text{and} \quad N_R^-(t) := \inf_{x \in B_{Rt} \cap (\partial \mathcal{H}_e^+ + te)} m_\mu(x, \mathcal{H}_e^-). \quad (3.21)$$

Lemma 3.8. *There exists $C = C(\bar{\mu}) > 0$ such that, for all $\mu \in (0, \bar{\mu}]$ and $R, t > 1$,*

$$\mathbb{E}|N_R^\pm(t) - \mathbb{E}[m_\mu(te, \mathcal{H}_e^-)]| \leq C a_\mu^{-1/2} t^{1/2} \log^{1/2}(1 + Rt). \quad (3.22)$$

Since $m_\mu(\cdot, \mathcal{H}_e^-)$ is approximately a constant on $B_{Rt}(te) \cap \{x \in \mathbb{R}^d : x \cdot e = t\}$ we compare m_μ with $m_\mu(\cdot, \mathcal{H}_e^- + te) + \mathbb{E}m_\mu(te, \mathcal{H}_e^-)$ in the domain $\mathcal{H}_e^+ + te$. This requires a quantitative and localized version of uniqueness for the planar metric problem. Then the error in linearity of $\mathbb{E}m_\mu(te, \mathcal{H}_e^-)$ can be rewritten as,

$$|\mathbb{E}(\mathbb{E}m_\mu(te, \mathcal{H}_e^-) + m_\mu(se, \mathcal{H}_e^- + te) - m_\mu((t+s)e, \mathcal{H}_e^-))|$$

where the quantity inside the expectation will be estimated by the comparison/uniqueness result.

For the argument we need the following refinement of the metric problem uniqueness, which is very close in spirit to finite speed of propagation.

Lemma 3.9. *Fix $e \in S^{d-1}$. Let $m^1 \in C(\mathcal{H}_e^+)$ and $m^2 \in C(\mathcal{H}_e^+)$ be respectively a sub-solution and a locally Lipschitz super-solution to $H(Dm, y) = \mu$ in \mathcal{H}_e^+ . Assume that there exist $M, K > 0$ and $R > L_\mu M$, L_μ being the upper bound for the Lipschitz constant of $r_\mu^{-1}(\cdot, x)$ in Lemma 3.2, such that,*

$$0 \leq m^1 \leq M + K|x|, \quad 0 \leq m^2 \leq M + K|x| \quad \text{and} \quad m_1 \leq m_2 \quad \text{on} \quad \partial \mathcal{H}_e^+ \cap B_R.$$

Then, for $s > 0$ such that $R \geq L_\mu M + (K+1)s$,

$$m_1(se) \leq m_2(se).$$

Now from Lemma 3.8 and Lemma 3.9 together we can prove the almost additivity (Lemma 3.7).

Proof of Lemma 3.7. Without loss we may assume that $s \leq t$. Let N^\pm as defined above in (3.22) with $R = L_\mu A_{\bar{\mu}} + (L_\mu + A_{\bar{\mu}})t$.

We apply Lemma 3.9 to $m^1(x) = m_\mu(x, \mathcal{H}_e^-)$ and $m^2(x) = m_\mu(x, \mathcal{H}_e^+ + te) + N^+(t)$ in the domain $\mathcal{H}_e^+ + te$. Checking the hypotheses of the localization result, we see that m^1 and m^2 are both solutions to

$$H(Dm^j, x) = \mu \quad \text{in} \quad \mathcal{H}_e^+ + te.$$

Recalling that $|Dm| = r_\mu(\frac{Dm}{|Dm|}, x) \leq A_\mu$, we obtain the Lipschitz estimate $\|Dm^j\|_\infty \leq A_\mu$ and, in particular,

$$0 \leq m^j(x) \leq A_\mu(t + |x - te|) \quad \text{in} \quad \mathcal{H}_e^+.$$

and then we take $M = K = A_\mu \leq A_{\bar{\mu}}$ in the statement of Lemma 3.9.

Finally, since the ordering $m^1 \leq m^2$ on $B_R(te) \cap (\partial \mathcal{H}_e^+ + te)$ follows directly from the definition of $N^+(t)$, we find using the particular our choice of R above,

$$m_\mu((t+s)e, \mathcal{H}_e^-) \leq m_\mu(se, \mathcal{H}_e^- + te) + N^+(t).$$

Taking expectations on both sides and using the \mathbb{R}^d -stationarity of $m_\mu(x, \mathcal{H}_e^+ + \cdot)$, we obtain

$$\begin{aligned} \mathbb{E}m_\mu((t+s)e, \mathcal{H}_e^-) &\leq \mathbb{E}m_\mu(se, \mathcal{H}_e^- + te) + \mathbb{E}m_\mu(te, \mathcal{H}_e^-) + C(\bar{\mu})a_\mu^{-1/2}t^{1/2}\log^{1/2}(1+Rt) \\ &= \mathbb{E}m_\mu(se, \mathcal{H}_e^-) + \mathbb{E}m_\mu(te, \mathcal{H}_e^-) + C(\bar{\mu})a_\mu^{-1/2}t^{1/2}\log^{1/2}(1+Rt) \\ &\leq \mathbb{E}m_\mu(se, \mathcal{H}_e^-) + \mathbb{E}m_\mu(te, \mathcal{H}_e^-) + C(\bar{\mu})a_\mu^{-1/2}(1+|\log L_\mu|)^{1/2}t^{1/2}\log^{1/2}(1+t). \end{aligned}$$

For the reverse inequality we compare $m^1(x) = m_\mu(x, \mathcal{H}_e^+ + te) + N^-(t)$ with $m^2(x) = m_\mu(x, \mathcal{H}_e^-)$. \square

We now return to the proof of Lemma 3.9.

Proof of Lemma 3.9. The t -level sets of the solution m_μ to the metric problem can be thought as the location at time t of a front moving with normal velocity r_μ^{-1} . Indeed, we show that $w(x, t) := m^1(x) \wedge t$ is a subsolution to

$$w_t + r_\mu \left(\frac{Dw}{|Dw|}, x \right)^{-1} |Dw| \leq 1 \quad \text{in } \mathcal{H}_e^+ \times (0, \infty). \quad (3.23)$$

Assuming for the moment (3.23) and noting that the assumptions yield that

$$w(\cdot, 0) \leq 0 \quad \text{in } \mathcal{H}_e^+ \quad \text{and} \quad w \leq m^1 \leq m^2 \quad \text{on } (\partial \mathcal{H}_e^+ \cap B_R) \times (0, \infty), \quad (3.24)$$

we make use of the finite speed of propagation property of the time dependent Hamilton-Jacobi equation with L_μ as in the statement to get

$$w(\cdot, t) \leq m^2 \quad \text{in } B_{R-L_\mu t} \cap \mathcal{H}_e^+.$$

Note that if $t \geq m^1(se)$, which follows from the assumption of the Lemma if $t = M + Ks$, then $m^1(se) = w(se, t)$, and we obtain that, as long as $R - L_\mu(M + Ks) \geq s$,

$$m^1(se) \leq m^2(se).$$

It remains to prove (3.23), which follows by showing that w is the uniform limit, as $\lambda \rightarrow \infty$, of the w_λ 's given by

$$w_\lambda(x, t) := -\frac{1}{\lambda} \log(\exp(-\lambda m^1(x)) + \exp(-\lambda t)), \quad (3.25)$$

which are themselves subsolutions of (3.23).

Since the limit part of the assertion above is obvious, here we only check the subsolution property. For this we argue as if the w_λ 's were smooth, the rigorous argument following from classical viscosity solutions considerations.

Let $Z(x, t) := \exp(-\lambda m^1(x)) + \exp(-\lambda t)$. It follows that

$$Dw^\lambda = Dm^1 Z^{-1} \exp(-\lambda m^1) \quad \text{and} \quad w_t^\lambda = Z^{-1} \exp(-\lambda t).$$

Then, using 1-homogeneity of the Hamiltonian and the subsolution property of m^1 , we get

$$\begin{aligned} w_t^\lambda + r_\mu \left(\frac{Dw^\lambda}{|Dw^\lambda|}, x \right)^{-1} |Dw^\lambda| &= Z^{-1} \exp(-\lambda t) + r_\mu \left(\frac{Dm^1}{|Dm^1|}, x \right)^{-1} |Dm^1| Z^{-1} \exp(-\lambda m^1) \\ &\leq Z^{-1} (\exp(-\lambda t) + \exp(-\lambda m^1)) = 1. \end{aligned}$$

□

Finally one obtains the rate of convergence of the expectations by using the almost additivity at a sequence of dyadic scales. This yields the first part of the Proposition 3.6.

Lemma 3.10. *There exists $\bar{r}_\mu(e) \geq 0$ such that, all $\bar{\mu} > 1$, $\mu \in (0, \bar{\mu}]$ and $t > 1$,*

$$\left| \frac{1}{t} \mathbb{E} m_\mu(te) - \bar{r}_\mu(e) \right| \leq C_\mu t^{-1/2} \log^{1/2}(1+t),$$

where $C_\mu := C(\bar{\mu}) a_\mu^{-1/2} (1 + |\log L_\mu|)^{1/2}$.

3.7. The effective Hamiltonian \bar{H} . Using \bar{r}_μ we define, for all $\xi \in \mathbb{R}^d \setminus \{0\}$, the effective Hamiltonian $\bar{H}(\xi)$ as

$$\bar{H}(\xi) := \inf \{ \mu > 0 : \bar{r}_\mu \left(\frac{\xi}{|\xi|} \right) \geq |\xi| \}. \quad (3.26)$$

We discuss the structural properties of \bar{r}_μ and \bar{H} which are inherited from the original Hamiltonian.

The continuity and monotonicity properties of \bar{r}_μ follow from the next lemma, which we state without proof.

Lemma 3.11. *For any $\bar{\mu} \geq 1$ there exists $C(\bar{\mu}) \geq 1$ such that, $\mu \in (0, \bar{\mu}]$ and $e, e_1, e_2 \in S^{d-1}$,*

$$C(\bar{\mu})^{-1} \mu \leq \frac{d}{d\mu} \bar{r}_\mu(e) \leq C(\bar{\mu}) \frac{1}{\mu \omega(\mu)} \quad \text{and} \quad |\bar{r}_\mu(e_1) - \bar{r}_\mu(e_2)| \leq C(\bar{\mu}) (1 + L_\mu) |e_1 - e_2|.$$

In the next lemma, we assert that the quantitative star-shapedness (3.5) of the sub-level sets of H are inherited by the \bar{H} . Since the latter may have flat parts, which are the zero level set $\{\xi : \bar{H}(\xi) = 0\}$, we no longer have strict monotonicity along radii. However, as soon as $\bar{H}(te)$ is positive, the strict monotonicity returns. The continuity properties of \bar{H} are more easily understood via the approximate cell problem which homogenizes. Since this is standard, we omit the statements.

Lemma 3.12. *There exists $c = c(\bar{\mu}) > 0$ such that, for all $\xi \in \{\xi \in \mathbb{R}^d : 0 < \bar{H}(\xi) \leq \bar{\mu}\}$,*

$$\frac{\xi}{|\xi|} \cdot D_\xi \bar{H}(\xi) \geq c(\bar{\mu}) \bar{H}(\xi) \omega(\bar{H}(\xi)).$$

In particular, the sub-levels of \bar{H} are star-shaped with respect to the origin.

Proof. Recall that, if $\mu = \bar{H}(te) > 0$, then $\bar{r}_\mu(e) = t$. For any $\nu > \mu$ let $\delta := C\nu^{-1}(\omega(\nu) \wedge \omega(\mu))^{-1}(\nu - \mu)$. It follows from the Lipschitz property of $\mu \rightarrow \bar{r}_\mu(e)$ that

$$\bar{r}_\nu(e) \leq \bar{r}_\mu(e) + \delta,$$

and, in view of the definition of \bar{H} ,

$$\nu \leq \bar{H}((t + \delta)e).$$

Thus

$$\frac{\bar{H}((t + \delta)e) - \bar{H}(te)}{\delta} \geq \frac{\nu - \mu}{\delta} = c(\bar{\mu})\nu(\omega(\nu) \wedge \omega(\mu)).$$

Taking the limit $\nu \rightarrow \mu$ and using the continuity of ω implies the desired lower bound. \square

3.8. Connection with the approximate correctors. Finally we describe the connection between the planar metric problem and the solution $v^\delta(x, \xi)$ of the approximate corrector problem,

$$\delta v^\delta + H(\xi + Dv^\delta, x) = 0 \quad \text{in } \mathbb{R}^d.$$

The aim is to show the convergence of $-\delta v^\delta(0, \xi) \rightarrow \bar{H}(\xi)$ stated in the next proposition. Since the star-shapedness property and uniqueness of the metric problem do not play any significant role here, we omit any discussion about proofs for which we refer to [1].

Proposition 3.13. *There exist $C = C(\bar{\mu})$, $c = c(\bar{\mu}) > 0$ depending on the constants from (3.6), (3.4) and (3.7) such that, for all $\xi \in \mathbb{R}^d$ such that $0 \leq \bar{H}(\xi) < \bar{\mu}$ and all $\lambda, \delta > 0$,*

$$\mathbb{P}(|\delta v^\delta(0, \xi) + \bar{H}(\xi)| > \lambda \omega'(\delta |\log \delta|)) \leq C \exp(-c\lambda^{4 \wedge d})$$

where the modulus ω' is the inverse of the modulus $\lambda \mapsto \lambda^4 / |C + \log(\lambda \omega(\lambda/2))|$ defined on $[0, 1]$.

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